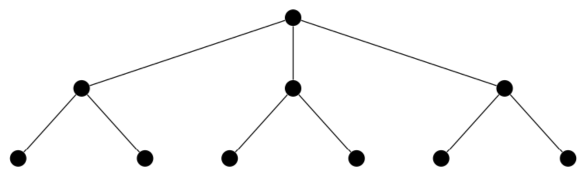
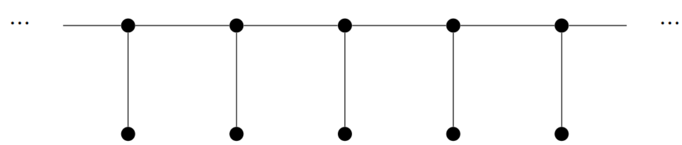
1 intro

I'll use the same notation as this paper by Bonato, Lehner, Marbach, and Nir as closely as I can. That means that for any tree T, the localization number of T is expressed with $\zeta(T)$. There are also a few precisely defined trees. There's $\hat{T}$, which looks like this:



As well as T_1^∞, which looks like this:



It is already known that any tree containing $\hat{T}$ or T_1^∞ has localization number 2. In fact, every finite tree with localization number 2 contains an instance of $\hat{T}$. However, it is not clear if every locally finite tree with localization number 2 contains a tree in $\{\hat{T}, T_1^\infty\}$. The exact question posed is:

Question 1. *An interesting problem is determining the minimal locally finite trees with countably (or even finitely) many ends and localization number 2. We think that examples other than $\hat{T}$ and T_1^∞ exist, but it is open whether there exists an infinite family of minimal locally finite trees with two ends and localization number 2.*

Or, rephrased

Question 2. *Does there exist a tree T with $\zeta(T) = 2$ that does not contain $\hat{T}$ or T_1^∞? Moreover, do such trees with 2 ends exist?*

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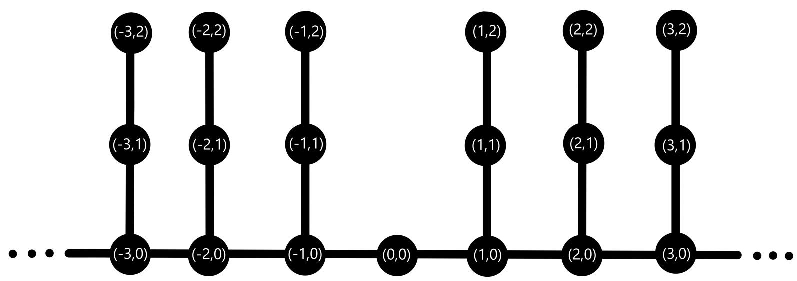
It's nice that the scope of the question is limited with that second question. I'll build larger and larger trees that do not contain $\hat{T}$ or T_1^∞, and see when we get a localization number above 1.

Let T be a 2-ended tree that does not contain T_1^∞ or $\hat{T}$. Because T does not contain T_1^∞, it must have a double ray with at least one vertex of degree 2. Label that vertex with degree 2 $(0, 0)$, and label the other points along that double ray with $(n,0)$. For any vertex V on T that is not on the double ray, label V with n, m if V is closer to $(n,0)$ than any other point on the double ray, and V is a distance of m from $(n,0)$.

For any $n \neq 0$, let N be a finite subtree rooted at $(n,0)$. If the cop ever knows that the robber is on N, then the robber can be found by the same method described in this paper by Seager. Therefore, I have the initial impression that a 2-ended graph with length 2 paths on each vertex $(n,0)$ with $n \neq 0$, as shown.



As defined, T has a localization number 1.

Lemma 1. *Let T be the tree with vertices uniquely labeled (n,m), with $n \in \mathbb{Z}$ and $m \in \{0,1,2\}$, except for $(0,1)$ and $(0,2)$, where $(n,0)$ shares an edge with $(n+1,0)$ for each value of n and $(n,1)$ shares edges with $(n,0)$ and $(n,2)$. If T has no other edges or vertices, then $\zeta(T) = 1$*

Proof. For this proof, I'll first show that the robber can be restricted to the vertices $\{(n,0),(n+1,1),(n+2,2)\}$ for positive n. Then, I'll show that this can be pushed to $\{(0,0),(1,1),(2,2)\}$. Finally, from there, the cop may locate the robber.

First, the cop may probe at $(0,0)$ and receive a distance from the robber of d. The robber set at that point is:

$$\{(-d,0),(-d+1,1),(-d+2,2),(d-2,2),(d-1,1),(d,0)\}$$

Next, the cop may probe at $(-d-1,0)$. The possible distances from that point to points in the extended robber set is:

distance from $(-d-1,0)$	new robber set
0	$(-d-1,0)$
1	$(-d,0)$
2	$(-d,1),(-d+1,0)$
3	$(-d,2),(-d+1,1),(-d+2,0)$
$2d$	$(d-2,1),(d-1,0)$
$2d+1$	$(d-2,2),(d-1,1),(d,0)$
$2d+2$	$(d-1,2),(d,1),(d+1,0)$

If the second probe gives a distance of 0 or 1, the robber is found instantly. If the second probe gives a distance of 2 or 3, then the robber has been restricted to vertices of the form $\{(-n,0),(-n-1,1),(n-2,2)\}$, which is identical to $\{(n,0),(n+1,1),(n+2,2)\}$ up to sign. If the second probe gives a distance above 3, then a third probe is required.

If a third probe is required, we have so far restricted the robber to one of three sets, whose union is:

$$\{(d-2,1),(d-1,0),(d-2,2),(d-1,1),(d,0),(d-1,2),(d,1),(d+1,0)\}$$

Therefore, let the cop put the third probe at $(d+2,0)$, giving the following robber sets:

distance from $(-d-1,0)$	new robber set
0	$(d+2,0)$
1	$(d+1,0)$
2	$(d,0),(d+1,1)$
3	$(d-1,0),(d,1),(d+1,2)$
4	$(d-2,0),(d-1,1),(d,2)$

Regardless of what distance is found in the third probe, the robber is restricted to a set of vertices $\{(n,0),(n+1,1),(n+2,2)\}$. Next, the cop may push the robber to $\{(n-1,0),(n,1),(n+1,2)\}$ using the following sequence of probes:

Robber set	Probe	$d=1$	$d=2$	$d=3$	$d=4$
$(n,0),(n+1,1)$ $(n+2,2)$	$(n+2,1)$	$(n+2,2)$	$(n+1,0)$	$(n,0),(n+1,1)$	$(n-1,0),(n,1)$ $(n+1,2)$
$(n,0),(n+1,1)$	$(n+1,2)$	$(n+1,1)$	$(n+1,0)$	$(n,0)$	$(n-1,0),(n,1)$

Once the cop has pushed the robber to $\{(0,0),(1,1),(2,2)\}$, the robber can be found with the following probing strategy:

Robber set	Probe	$d=1$	$d=2$	$d=3$	$d=4$
$(0,0),(1,1),(2,2)$	$(0,0)$	$(-1,0),(1,0)$	$(1,1)$	$(1,2),(2,1)$	$(2,2)$
$(1,2),(2,1)$	$(1,1)$	$(1,2)$		$(2,1)$	$(2,2)$
$(-1,0),(1,0)$	$(-2,0)$	$(-1,0)$	$(-1,1),(0,0)$	$(1,0)$	$(1,1),(2,0)$
$(-1,1),(0,0)$	$(-1,2)$	$(-1,1),$	$(-1,0)$	$(0,0)$	$(1,0)$
$(1,1),(2,0)$	$(3,0)$	$(2,0),$	$(2,1),(1,0)$	$(1,1)$	$(1,2)$
$(2,1),(1,0)$	$(2,2)$	$(2,1),$	$(2,0)$	$(1,0)$	$(0,0),(1,1)$
$(0,0),(1,1)$	$(1,2)$	$(1,1),$	$(1,0)$	$(0,0)$	$(-1,0)$

□

Unfortunately, this proof doesn't work if the cop never probes vertices off the double ray. This means that the translation of this proof to versions of T with different subtrees coming off the double ray require closer inspection. So let's take this same proof concept and expand it to larger trees.

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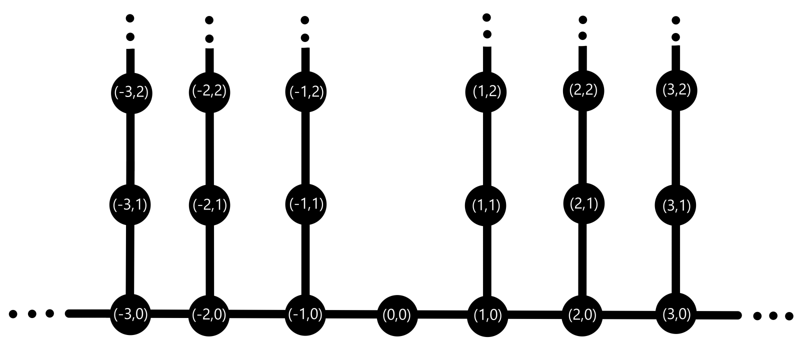
As promised, I'll extend the previous theorem to a larger tree. Let T be a 2-ended tree with vertices uniquely labelled with (n, m), with $n \in \mathbb{Z}/0$ and $m \in \mathbb{Z} \cap [0, h]$ where h is any natural number, as well as a vertex labelled $(0, 0)$. The tree T will also contain edges between the vertices $(n, 0)$ and $(n + 1, 0)$ for any n, and an edge between vertices (n, m) and $(n, m + 1)$ for any n, m. For convenience, this is illustrated here:



In other words, each vertex on the double ray has an arbitrarily long path coming off it. This graph will be shown to have localization number 1 using the same method used previously.

Lemma 2. *The cop is able to restrict the robber to a set of vertices* $\{(n, 0), (n+1, 1), (n + 2, 2)\}$ *for positive* n.

Proof. The cop may probe first at $(0, 0)$, and receive a distance of d. If they then probe at $(-d, 0)$, it will be clear if the robber is located at a vertex with a positive or negative x-coordinate. Without loss of generality, assume the robber is determined to lie to the right of $(0, 0)$.

The cop will then probe once to the left of the robber, then once to the right. The cop may probe at $(0, 0)$ and receive a distance of d_1. The robber set at that point consists of all vertices $(d_1 - i, i)$ where i is a natural number less than h and d_1. The extended robber set consists of the vertices $(d_1 - i + \psi, i)$, where $\psi \in [-1, 0, 1]$.

Next the cop may probe to the right of the robber, specifically at $(d_1 + 1, 0)$, and receive distance d_2. If d_2 is 0 or odd, the robber is instantly found, so assume d_2 is even. If $d_2 = 2$, then the robber is restricted to $\{(d_1 - 1, 0), (d_1, 1)$, satisfying the lemma statement. So assume now that $d_2 \geq 4$ and that the robber

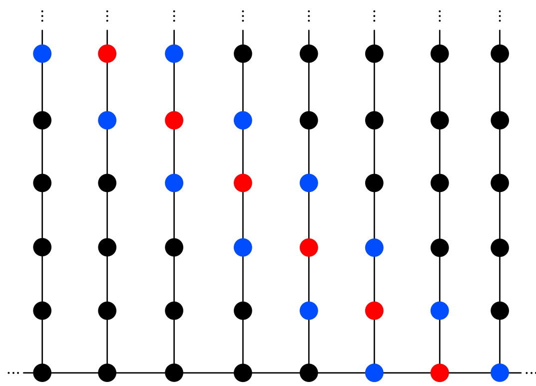


Figure 1: The situation after the cop gets a distance of d_1. Here, the robber set is colored in red, and the extended robber set is colored in blue.

is restricted to some set of vertices $\{(n,m),(n+1,m+1)\}$. I'll illustrate the options for this as well.

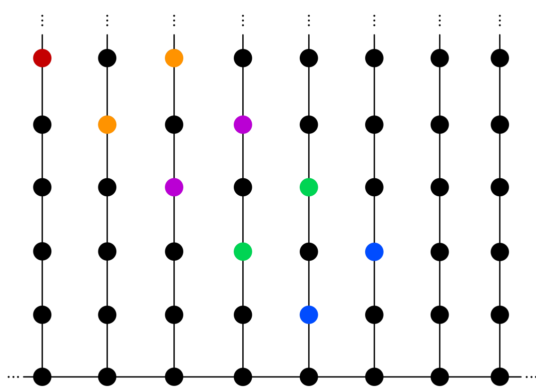
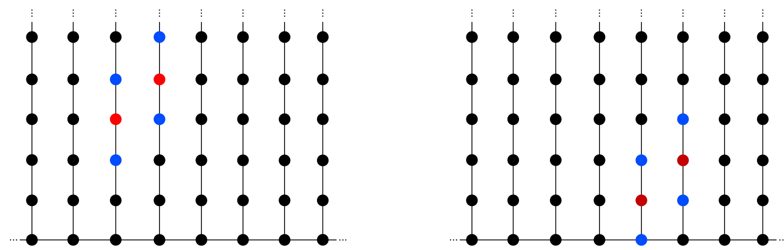


Figure 2: The situation after the cop gets a distance of d_2. Here, the different robber sets are distinguished by color.

As mentioned, the robber is now restricted to a set of 2 vertices $\{(n,m),(n+1,m+1)\}$. If $m>1$, then that set is raised slightly off the double-ray, giving an extended robber set of $\{(n,m+\psi),(n+1,m+1+\psi)\}$ for all values of $\psi \in [-1,0,1]$. The cop will now probe at $(n,1)$

If $m>1$, then probing at $(n,1)$ instantly reveals the location of the robber, as shown in the following table:



(a) $m > 1$ (b) $m = 1$

Figure 3: The regular and extended robber set for $m > 1$ and $m = 1$

distance from $(n, 1)$	robber location
$m - 2$	$(n, m - 1)$
$m - 1$	(n, m)
m	$(n, m + 1)$
$m + 1$	$(n + 1, m)$
$m + 2$	$(n + 1, m + 2)$
$m + 3$	$(n + 1, m + 2)$

If $m = 1$, then the robber is restricted to a pair of vertices labeled $(n, 1)$ and $(n + 1, 2)$. The cop can either locate the robber or shift the robber to vertices labeled $(n - 1, 0)$ and $(n, 1)$ with the following two moves:

Robber set	$(n, 1), (n + 1, 2)$	$(n, 0), (n, 2)$
probe	$(n, 1)$	$(n + 1, 0)$
$d = 0$	$(n, 1)$	$(n + 1, 0)$
$d = 1$	$(n, 0), (n, 2)$	$(n, 0)$
$d = 2$		$(n, 1), (n - 1, 0)$
$d = 3$	$(n + 1, 1)$	$(n, 2)$
$d = 4$	$(n + 1, 2)$	$(n, 3)$
$d = 5$	$(n + 1, 3)$	

□