Mosers worm notes

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1 Introduction

These are sloppy and casual notes that I'm making to collect my thoughts on Mosers worm problem. First, here is an important definition:

Definition 1. A set of points (in euclidean 2D space) A *accommodates* a set B when there is a set B' such that $B' \subseteq A$, and B' is congruent to B.

Equivalently, a set of points A *accommodates* a set B when there is a set A' such that $B \subseteq A'$, and A' is congruent to A.

For these notes, A accommodates B will be notated with $B \sqsubseteq A$.

From here on out, I might be a little bit looser, and not worry so much about the idea that S' is appreciably different from S. Each shapes name might also be loosely used to apply any translation and rotation of that shape.

Problem 1. *What is the smallest convex shape that can accommodate any curve of length 1?*

I have three approaches to this problem I'm considering. I'm going to write this out in the order they're occurring to me.

2 Polygonal chains

This problem has been unsolved since 1966. I therefore conclude that it is very difficult. Here is a slightly simpler problem:

Problem 2. *What is the smallest convex shape that can accommodate any polygonal chain of length 1?*

As a matter of fact, I'm confident that this second problem is effectively equivalent to the first one. I worked on a similar problem to this with my buddy Emma Joe back in undergrad. So, here's an even easier version of the problem:

Problem 3. *What is the smallest convex shape that can accommodate any 3-segment polygonal chain of length 1?*

I try not to look to much at work that's already been done on a problem while I'm just beginning to think about it myself, but it looks like

1. None of these problems have been solved (I think :P)

2. Problem 3 has been shown not to imply Problem 4, but Problem 2 does

Okay, from here out, I'm not going to read anything other people have written on the problem, at least until I feel like I've done what I can.

I don't think it'll cause any harm to come up with some notation, if it's just for my use. Given a set of points S, the convex hull of S will be given by $[S]$. That seems reasonable, and I don't foresee that causing confusion.

Okay, let me wave my hands just a little bit. I don't have a written proof for this, abut it seems safe to just state.

Observation 1. Given a sets A and B, with A being convex, $B \sqsubseteq A$ if and only if $[B] \sqsubseteq A$

This can be proven, but I don't want to type the proof out right now. Sorry. The proof is left as an exercise for the reader, i guess.

Okay, so, any shape that accommodates everything has to accommodate the convex hulls of 3-link polygonal chains. Therefore, let's start by trying to accommodate polygonal chains whose convex hulls are rectangles.

Definition 2. Let R be the set of 3-link unit-length polygonal chains, for which the first and third link have the same length, and all corners are $90°$ clockwise.

Observation 2. The convex hull of any curve in R is a rectangle with length ℓ and width w, with $\ell + 2w = 1$

So, we want to accommodate all these rectangles. The two hardest shapes in this set are the square with sides of length $\frac{1}{3}$, and the line of length 1. Here's a useful definition we can use:

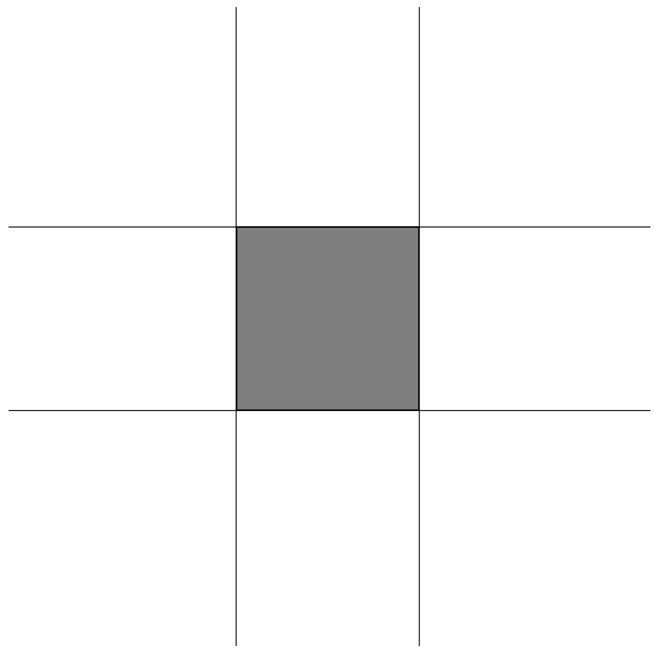
Definition 3. For short, we'll refer to the convex hull of the union of a unit-length line and a square with side length $\frac{1}{3}$ as a *SLCH*, short for "Square-line convex hull".

If a SLCH cannot accommodate a distinct, smaller SLCH, we'll call that a *minimal* SLCH.

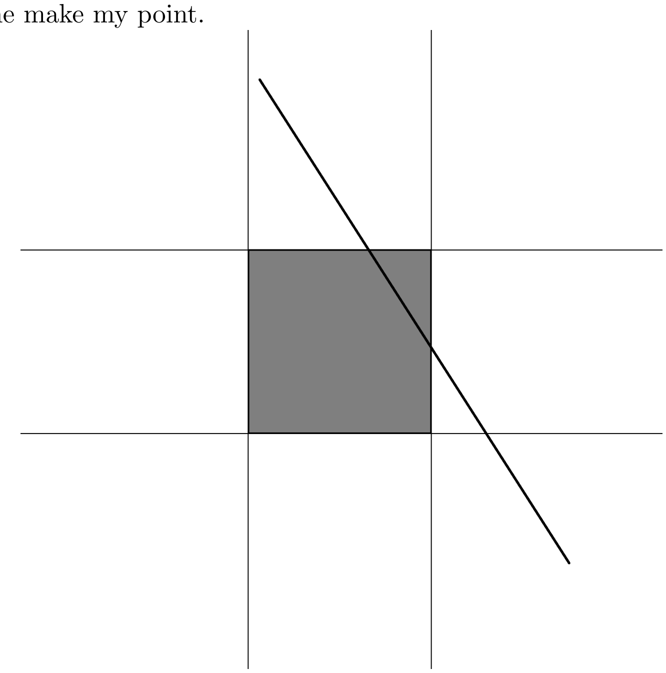
Definition 4. A shape that can accommodate any unit length curve will be called a *blanket*.

Observation 3. Any blanket must accommodate a minimal SLCH.

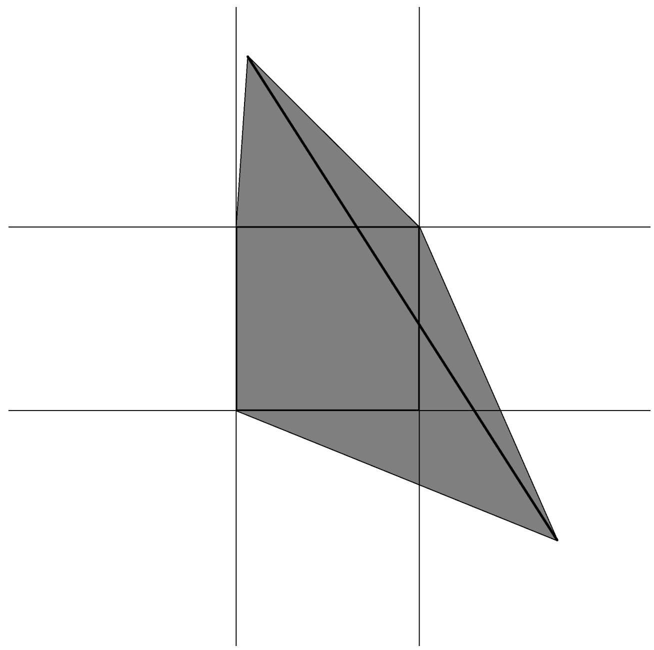
I'm very interested in figuring out what the smallest minimal SLCH is. We can start with the square. I'll include some lines, to make things easier to understand.



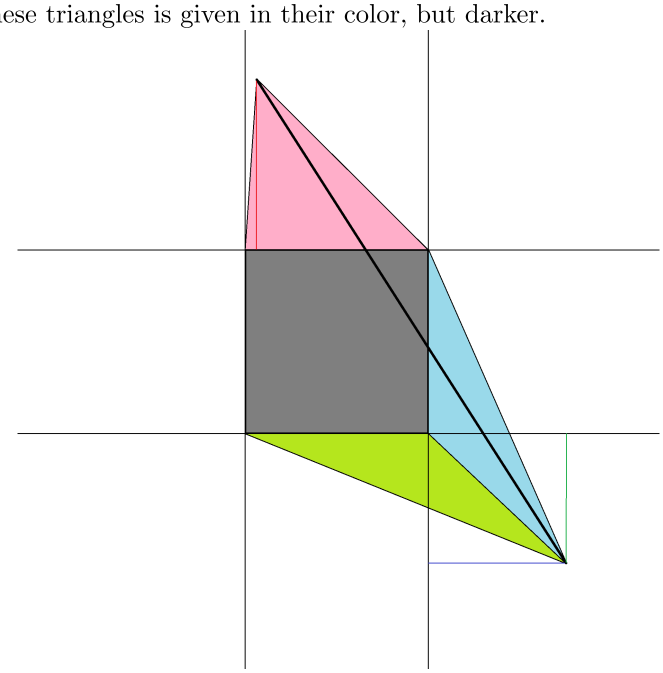
Then, the line has to be included. I'll just drop it down in a way that helps me make my point.



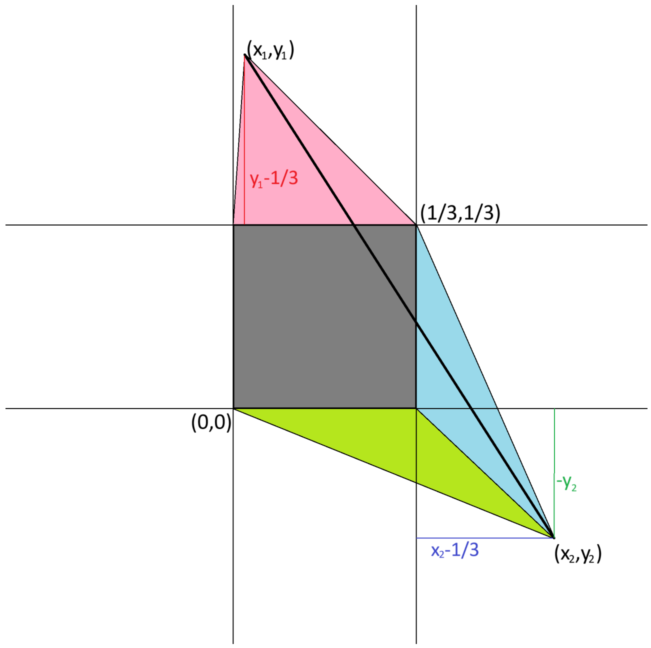
The convex hull looks like this:



And we can divide the convex hull into triangles. The altitude of each of these triangles is given in their color, but darker.



Let's finish off by labeling some of the points and distances involved.



Now I can finally make my point! If the endpoints of the line are located at (x_1, y_1) and (x_2, y_2), then there are at most 4 triangles added to the overall shape (actually, you could have more if the line doesn't intersect with the square, but dw about it). If an endpoint of the line ends in one of the areas above or to the side of the square, one triangle is added, and if it's in one of the corner sections, you add two triangles.

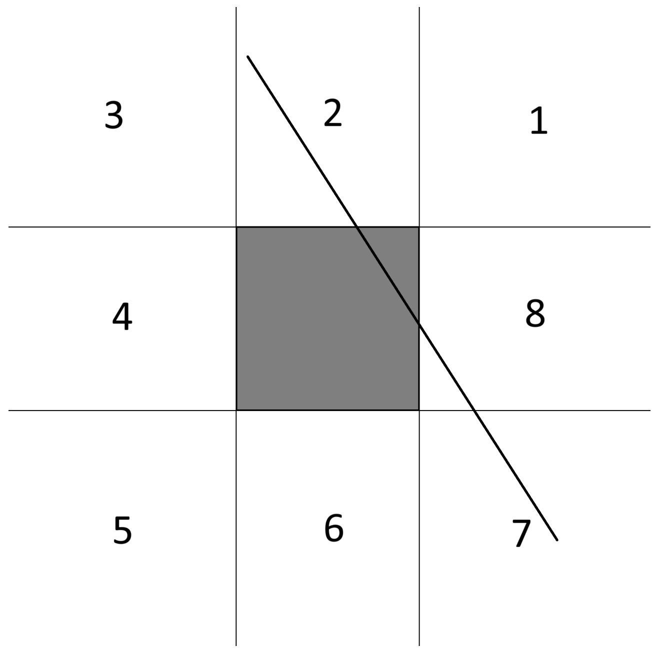
If a is an x or y coefficient of an endpoint of the line, then the triangle added has a base length of $\frac{1}{3}$, and the area A of the triangle added is given by:

$$A = \begin{cases} \frac{-a}{6} & a < 0 \\ 0 & 0 \le a \le \frac{1}{3} \\ \frac{\left(a - \frac{1}{3}\right)}{6} & a > \frac{1}{3} \end{cases}$$

The two endpoints of the triangle are 1 unit distance apart. Therefore:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 1$$

That piece of this problem about the area being a piecewise thing is a bit of a disaster though. I'll give some names to the regions around the square.



First, I'll mention a few things.

Statement 1. If one end of the line is inside the square, then the SLCH is not minimal.

This is because, in any case, you can slide the line deeper into the square to get a smaller SLCH.

Statement 2. If the line does not pass through the square, then the SLCH is not minimal.

This is because you could translate either the line or the square towards the other to get a smaller SLCH. I know that neither of these arguments are rigorous, but I've convinced myself in my paper notes.

Ok, let's go back to considering what the smallest possible SLCH is. Considering the possible placement of the line with respect to the square, there are 3 possibilities.

1. (x_1, y_1), and (x_2, y_2) are in opposite odd numbered sections.
2. (x_1, y_1), and (x_2, y_2) are on opposite even numbered sections.
3. (x_1, y_1) is in an odd numbered section, and (x_2, y_2) is in an even numbered section

Without loss of generality, we may express these three options like this:

1. (x_1, y_1) is in section 5, and (x_2, y_2) are in opposite corners.

2. (x_1, y_1), and (x_2, y_2) are on opposite sides of the square, such as directly above and under.

3. (x_1, y_1) is in a corner and (x_2, y_2) is straight to one side of the square.

for situation 1, let's assume wlog that $x_1, y_1 \leq 0$, and that $x_2, y_2 \geq \frac{1}{3}$. So, the area added to the shape is given by:

$$A = \frac{-x_1}{6} - \frac{y_1}{6} + \frac{x_2 - \frac{1}{3}}{6} + \frac{y_2 - \frac{1}{3}}{6}$$
$$6A = x_2 + y_2 - \frac{2}{3} - x_1 - y_1$$

As mentioned, the distance between the two points is defined with:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 1$$

Let's say we want to find the best location for (x_2, y_2), assuming (x_1, y_1) is stationary. This can be done by expressing y_1 in terms of the other variables:

$$1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$1 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$(y_2 - y_1)^2 = 1 - (x_2 - x_2)^2$$
$$y_2 - y_1 = \sqrt{1 - (x_2 - x_1)^2}$$
$$y_2 = y_1 + \sqrt{1 - (x_2 - x_1)^2}$$

Plugging this into the previous area equation gives:

$$6A = x_2 + y_1 + \sqrt{1 - (x_2 - x_1)^2} - \frac{2}{3} - x_1 - y_1$$
$$6A = x_2 + \sqrt{1 - (x_2 - x_1)^2} - \frac{2}{3} - x_1$$
$$6A = x_2 + \sqrt{1 - x_1^2 + 2x_1x_2 - x_2^2} - \frac{2}{3} - x_1$$

To find the minima of this function, we'll take the derivative of the area:

$$
\begin{aligned}
\frac{d}{dx_2}(6A) &= 1 + \frac{2x_1 - 2x_2}{2\sqrt{1 - (x_2 - x_1)^2}} \\
\frac{d}{dx_2}(6A) &= 1 + \frac{x_1 - x_2}{\sqrt{1 - (x_2 - x_1)^2}}
\end{aligned}
$$

When is $\frac{d}{dx_2}(6A)$ equal to 0?

$$
\begin{aligned}
0 &= 1 - \frac{x_1 - x_2}{\sqrt{1 - (x_2 - x_1)^2}} \\
1 &= \frac{x_1 - x_2}{\sqrt{1 - (x_2 - x_1)^2}} \\
\sqrt{1 - (x_2 - x_1)^2} &= x_1 - x_2 \\
1 - (x_2 - x_1)^2 &= (x_1 - x_2)^2 \\
1 &= 2\,(x_1 - x_2)^2 \\
\frac{1}{4} &= x_1 - x_2
\end{aligned}
$$

This only takes a value of 0 outside it's domain, implying that its minima are only at the boundaries.

3 Taut curves

Okay, I might be losing it here just a touch, but I'm gonna let myself cook. These are sloppy and casual notes that I'm making to collect my thoughts on Mosers worm problem. First, here is an important definition:

Definition 5. A set of points (in euclidean 2D space) A *accommodates* a set B when there is a set B' such that $B' \subseteq A$, and B' is congruent to B.

Equivalently, a set of points A *accommodates* a set B when there is a set A' such that $B \subseteq A'$, and A' is congruent to A.

For these notes, A accommodates B will sometimes be notated with $A \sqsupseteq B$. If B does not accommodate A, then we write $A \sqsupset B$

From here on out, I might be a little bit looser, and not worry so much about the idea that S' is appreciably different from S. Each shapes name might also be loosely used to apply any translation and rotation of that shape.

Problem 4. *What is the smallest convex shape that can accommodate any curve of length 1?*

My goal here is to see what curves we don't need to worry about. There's two prongs to this:

- Show that curve A is always accommodated when curve B is accommodated.
- Show that when curves A and B are accommodated, curve C is also accommodated.

To that end, we give these definitions:

Definition 6. Given a shape A, the convex hull of A will be noted $[A]$.

Definition 7. The length of a curve X will be noted ℓX. The section of an open curve X between two points a and b will be noted with X_a^b, so that the length of an open curve between between two points a and b will be noted with ℓX_a^b.

For any set of points $p_1, p_2 \ldots$, the polygonal chain that connects each of them in order will be denoted $\langle p_1, p_2 \ldots \rangle$.

Definition 8. A unit-length curve X is *taut* if there does not exist a curve X' such that $X' \sqsupset X$.

Immediately there are some lemmas we can form regarding taut curves without any further context. For example, this is the lemma that inspired the choice of the word "taut".

Lemma 1. *If a curve experiences any non-zero curvature at a point that is not on the border of its convex hull, it is not taut.*

Proof. Let X be a unit-length curve, and let x be a contiguous section of X that does not touch the border of $[X]$, except at two endpoints, a and b. Then let X' be identical to X, except that x is replaced with a straight line from a to b. The length of X' is necessarily less than 1, although X' has the same . Finally, let X'' be the unit-length curve that includes all of X', but with an added length at the end. X'' must have a convex hull that includes everything in $[X]$ and more. □

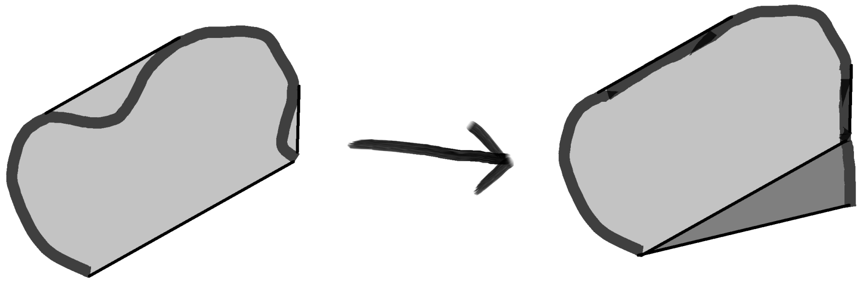


Figure 1: Hopefully, this drawing explains why I chose the word "taut" to describe this property. The curve on the left is "taut", in the sense that it could be stretched to coincide with the border of its convex hull, and that extra length could be used to create a larger hull. The light grey area is the same between the two shapes, but the dark grey area is added by using length more efficiently

Here's another brief and straightforward lemma about taut curves.

Lemma 2. *No self intersecting curves are taut.*

Let's start our effort to characterize taut curves with yet another definition

Definition 9. A *Type 1* curve is a curve for which all points on the curve lie on the boundary of its convex hull.

A *Type 2* curve is a curve for which there is one single segment that crosses the inside of the convex hull.

In general, a *Type n* curve is one with n distinct, discontinuous segments along the boundary of its convex hull, and $n-1$ distinct segments passing through the interior of the convex hull

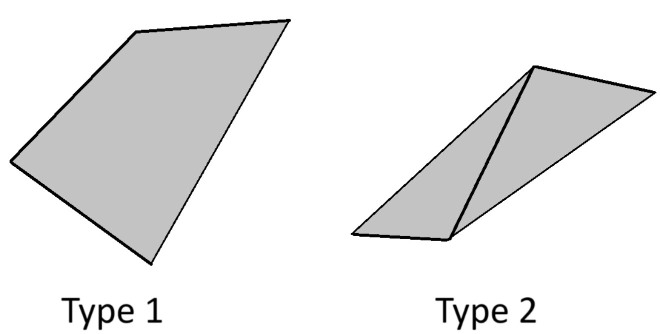


Figure 2: An example of a Type 1 and Type 2 curve.

Type 1 and Type 2 curves can also be thought of in terms of path direction. A path along a Type 1 curve moves entirely either clockwise or counterclockwise across the edge of its convex hull. A path along a Type 2 curve starts moving either clockwise or counterclockwise along the border, but then moves in the opposite direction for a length.

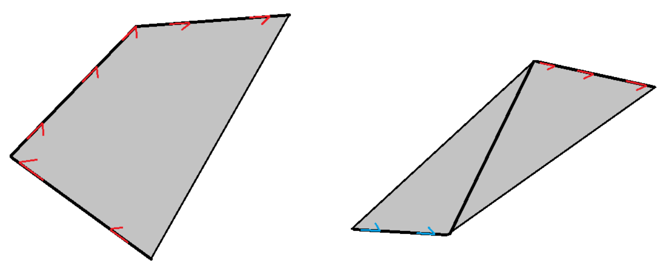


Figure 3: In this diagram, the path along the Type 1 curve moves only clockwise along the boundary of the convex hull. The path along the Type 2 curve begins traveling counterclockwise, but then changes to traveling clockwise.

Here's a brief lemma about Type n curves that will be useful in several results coming up

Lemma 3. *Let X be a taut curve, and let a and b be points on X such that X_a^b consists only of internal points of $[X]$. X_a^b has zero curvature.*

Proof. Suppose, for the sake of contradiction, that X_a^b does not have zero curvature. Then $\ell X_a^b > \ell\langle a, b\rangle$. Let X' be formed by replacing X_a^b in X with $\langle a, b\rangle$. It is clear that $\ell X' < X$, and so X is accommodated by X', and is not taut. □

I've only talked about Type 1 and Type 2 curves so far. Why hasn't any special attention been given to curves of other types? As it turns out, these curves are never taut.

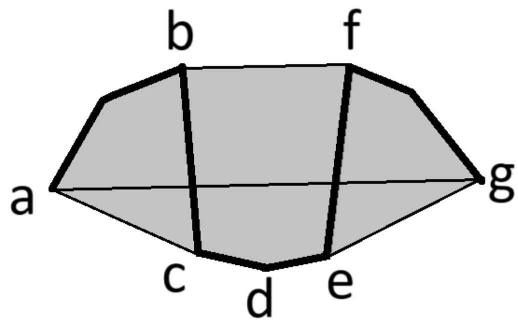
Theorem 1. *There are no taut Type n curves, for $n \geq 3$.*

Proof. To force a contradiction, let X be such a curve. Let a, b, c, d, e, f, and g be points on X, in that order, with the added requirements that:

- a, b, c, d, e, f, and g are all on the boundary of $[X]$.
- a and g are the endpoints of X.
- d is the point furthest from $\langle a, g\rangle$,
- $\langle b, c\rangle$ is internal to the convex hull of X.

- $\langle e, f \rangle$ is internal to the convex hull of X.

Here's a drawing to explain.



For ease of upcoming calculations, let $\langle a, g \rangle$ lie on the x axis, with a existing at $0, 0$.

Let ϕ be the minimal distance from d to $\langle a, g \rangle$. Suppose:

$$\frac{\ell X_b^f}{2} \leq \phi$$

In this case, define a' and g' to be these two points:

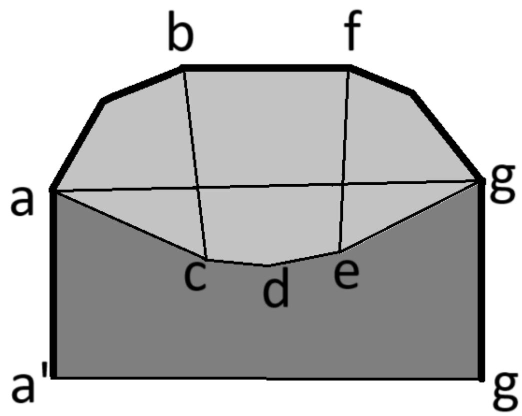
$$a' = \left(0, -\frac{\ell X_b^f}{2}\right)$$

$$g' = \left(\ell\langle a, g \rangle, -\frac{\ell X_b^f}{2}\right)$$

Then we can define X' to be the curve

$$X' = \langle a', a \rangle \cup X_a^b \cup \langle b, f \rangle \cup X_f^g \cup \langle g, g' \rangle$$

This is given in the drawing below:



Because a' and g' are both below d, the entirety of X must be inside X'

□

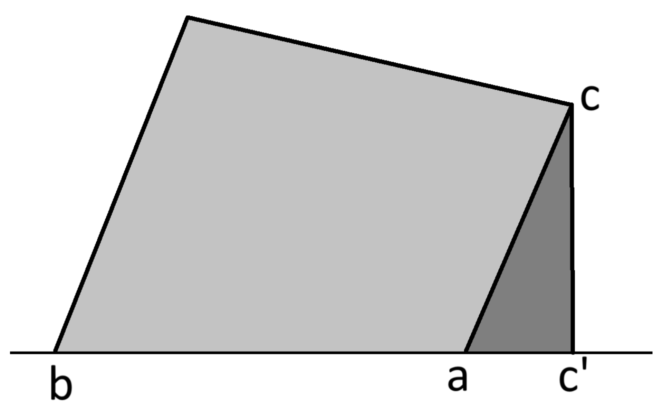
Let's limit a bit further which curves can be Type 1.

Lemma 4. *Let X be a Type 1 taut curve with endpoints a and b. Every point on X lies on a line perpendicular to $\langle a, b\rangle$.*

Proof. Let X be a curve as described above, and let c be the point on X that lies furthest from any line perpendicular to $\langle a, b\rangle$. without loss of generality, assume c is closer to a than it is to b.

Let α be the line that includes the line segment $\langle a, b\rangle$, and let c' be the point on α closest to c. Let X' be a curve identical to X, except that the section between c and a is replaced with a straight line between c and c'.

This situation is illustrated below:



It is clear that

$$\ell\langle c, c'\rangle < \ell C_c^a$$

and therefore

$$\ell X' < \ell X$$

Therefore, X' has a length less than X, and so is a unit curve. The convex hull of X' can accommodate X, but is larger. By definition, X is not taut. □

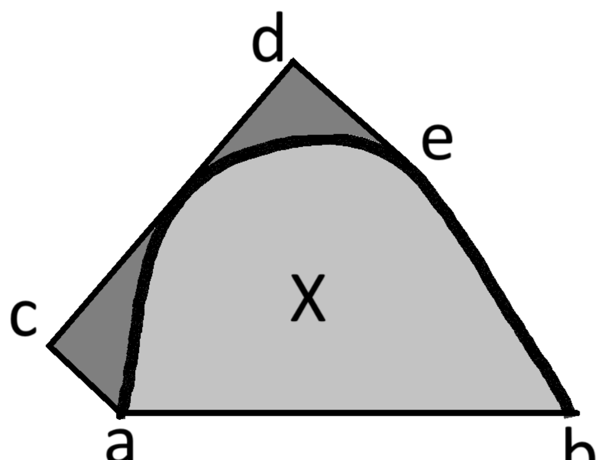
Here is a theorem that helps to characterize Type 1 curves:

Theorem 2. *A curve X with endpoints a and b is not accommodated by a distinct Type 1 curve if and only if:*

- *All points on X are on its convex hull.*
- *Every point on X lies on a line perpendicular to a point on $\langle a, b \rangle$.*
- *There does not exist a set of points $\{c, d, e, f\}$ such that:*
  - *$\langle d, e \rangle$ is tangent to X.*
  - *f is one of the end points of X.*
  - *e is a point on X that is not an end point.*
  - *$\langle c, d \rangle$ is perpendicular to $\langle f, c \rangle$ and $\langle d, e \rangle$*
  - *$\ell\langle f, b \rangle + \ell\langle c, f \rangle + \ell\langle e, d \rangle \leq \ell X_f^e$*

Proof. The fist of these two requirements are implied by the Lemma above. Therefore, this proof focuses on the second requirement

First, suppose that a set of points as described exists. Without loss of generality, let $f = a$. As before, I'll include a diagram to clarify what the situation described in the lemma statement is.

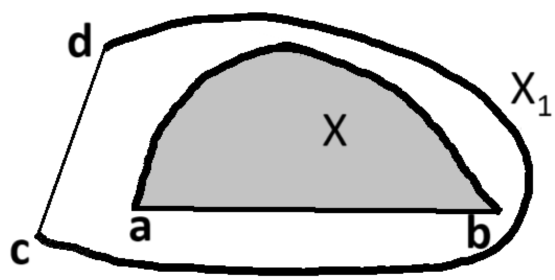


Let $X' = \langle c, a, b \rangle \cup X_b^e \cup \langle e, d \rangle$. As evident from the diagram, the convex hull of X' accommodates the convex hull of X. Next, consider the inequality assumed earlier:

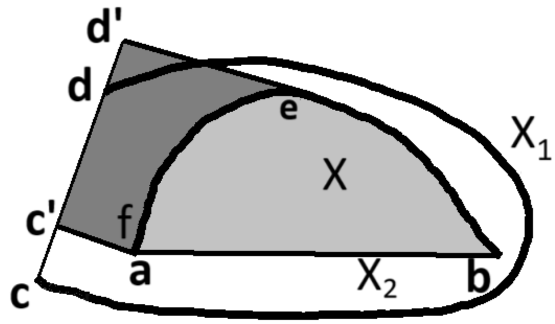
$$\begin{aligned}
\ell\langle a, b\rangle + \ell\langle c, a\rangle + \ell\langle e, d\rangle &> \ell X_a^e \\
\ell\langle c, a, b\rangle + \ell\langle e, d\rangle &> \ell X_a^e \\
\ell\langle c, a, b\rangle + \ell X_b^e + \ell\langle e, d\rangle &> \ell X_a^e + \ell X_b^e \\
\ell X' &> \ell X
\end{aligned}$$

Therefore, X' accommodates X.

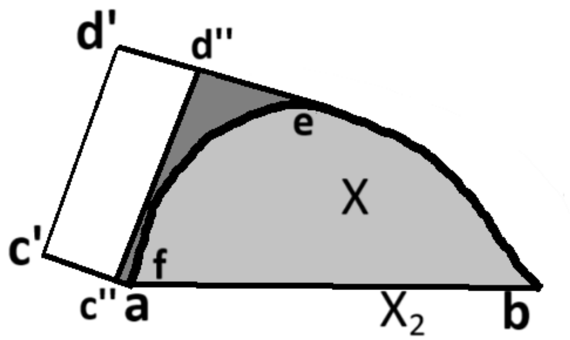
Next, suppose that such a set of points does not exist, but that X_1 is a Type 1 curve such that $[X_1] \sqsupset X$. Let c and d be the endpoints of X_1. An example of what this might look like is shown below:



Next, let α be the line passing through c and d. let X_2 be the shortest curve with endpoints on α that accommodates X. X_2 must consist of two lines parallel to α, as well as a section of the boundary of $[X]$, which may or may not include $\langle a, b\rangle$. Let c' and d' be the endpoints of X_2, and let e and f be the points where X_2 meets the convex hull of X, as shown below:



Finally, the length of X_2 can be reduced even further. Let d'' and c'' be points on $\langle d', e\rangle$ and $\langle c', e\rangle$ that are both a distance ϕ away from d' and c', with ϕ having the largest possible value without $\langle d'', c''\rangle$ intersecting the interior of $[X]$.



Let X_3 be X_2, without the lines from d' to d'' and from c' to c''. If f and e are both endpoints of X, then X_3 must contain the entirety of X. This is impossible, as $\ell X = 1$, so either e or f is not an endpoint of X.

Instead, assume (for the sake of contradiction), that neither f nor e are endpoints of X. This would require that at least one of the two points is not on a line parallel to $\langle a, b\rangle$.

Therefore, $\{c'', f, e, d''\}$ satisfies the requirements of the lemma. □

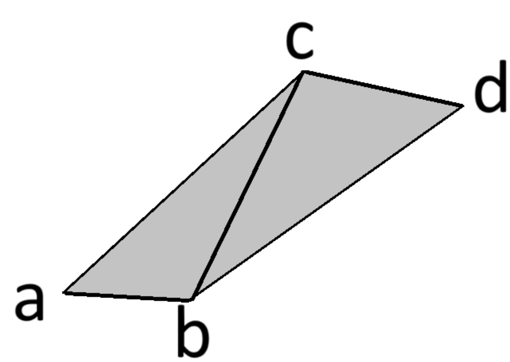
The reason I set out this lemma is because I want to demonstrate that there is a limit on the "height" of Type 1 curves. If $\langle a, b\rangle$ is very short, and X has a point very distant from $\langle a, b\rangle$, it would be easy to find a way to maximize ℓX_a^e and minimize $\ell\langle a, c\rangle$ and $\ell\langle d, e\rangle$. In short, This can be thought of as a generalized version of Lemma **??**.

It would be pretty cool to strengthen this theorem even further. Either way, it's time to start thinking about Type 2 curves.

Lemma 5. *Let X be a taught Type 2 curve, and let a and d be the end points of X. Let b and c be the endpoints of the section of X which are internal to the convex hull of X. It it required that:*

$$\ell\langle b.c\rangle \leq \ell\langle a, c\rangle$$
$$\ell\langle b.c\rangle \leq \ell\langle b, d\rangle$$

Proof. The locations of a though d are explained by this diagram:



Suppose for the sake of contradiction that $\ell\langle a.c\rangle > \ell\langle b, c\rangle$. Define X' such that

$$X' = X_b^a \cup \langle a, c\rangle \cup X_c^d$$

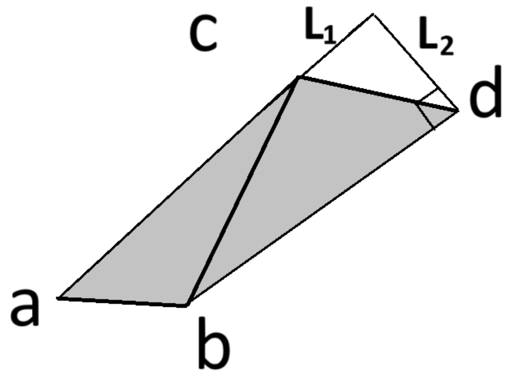
Because $\ell\langle a, c\rangle > \ell\langle b, c\rangle$, $\ell X' < \ell X = 1$. Then let X'' be identical to X', except that an extra bit is added to increase the area of the convex hull while keeping $\ell X'' \leq 1$. By definition, X is not taut.

By symmetry, X is also not taut if $\ell\langle b.d\rangle > \ell\langle b, c\rangle$ □

The non-internal sections of a Type 2 curve behave something like Type 1 curves themselves. For example, consider the following Lemma:

Lemma 6. *Let X be a taut Type 2 curve with endpoints a and d, and let b and c be the endpoints of the section of X that is internal to $[X]$. Let L_1 be the line passing through a and c, and let L_2 be the line perpendicular to $\langle b, d\rangle$ that passes through d. Then any point p on X_c^d must lie in the triangle bound by L_1, L_2, and $\langle c, d\rangle$.*

Proof. First, for clarity, a diagram of these lines is shown:



First, if p is a point on X_c^d that is on the "wrong" side of L_1, then c is an internal point of $[X]$. This is not true by assumption.

The point p must be on the exterior of $[X]$, by assumption, and so cannot be on the wrong side of $\langle c, d \rangle$.

Finally, suppose for the sake of contradiction that p lies on the wrong side of L_2. Without loss of generality, let p specifically be the point furthest from L_2. Then let p' be the point on the line containing b and d that is closest to p. If $X' = X_a^p \bigcup \langle p, p' \rangle$, then X' is shorter than X. Because $\langle p, p' \rangle$ is perpendicular to $\langle p', d \rangle$, p' must also be further from L_2 than any other point in X. Therefore, X' accommodates X, giving a contradiction.

□

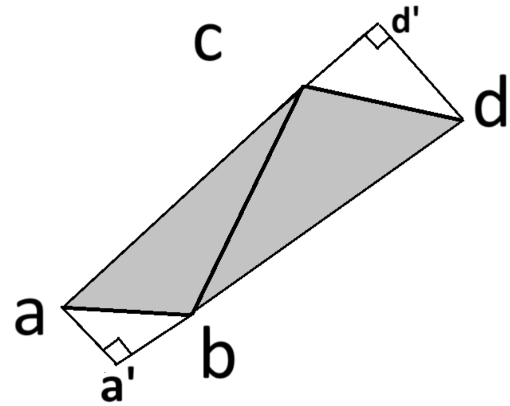
This lemma comes with a few charming corollaries:

Corollary 1. *Let X be a Type 2 curve with endpoints a and d, and let b and c be the endpoints of the section of X that is internal to $[X]$. Let L_2 be the line perpendicular to $\langle b, d \rangle$ that passes through d. If p and q are points on X_c^d such that $\ell X_q^d < \ell X_p^d$, then q is not further from L_2 than p is.*

Corollary 2. *Let X be a Type 2 curve with endpoints a and d, and let b and c be the endpoints of the section of X that is internal to $[X]$. The angles formed by $\langle b, d, c \rangle$ and $\langle c, a, b \rangle$ are acute.*

Finally, we can make a lemma effectively characterizing taut Type 2 curves.

Lemma 7. *Let X be a taught Type 2 curve, and let a and d be the end points of X. Let b and c be the endpoints of the section of X which are internal to the convex hull of X. Let a' be the point that is co-linear to a and c, such that $\langle a, a', b \rangle$ is a right triangle. Similarly, let d' be the point that is co-linear to b and d, such that $\langle c, d', d \rangle$ is a right triangle. This is shown in the illustration below.*



If X is taught, then:

- $\ell\langle b.c\rangle \leq \ell\langle a, c\rangle$
- $\ell\langle b.c\rangle \leq \ell\langle b, d\rangle$
- *Every point on X between c and d lies in the triangle bounded by L_1, L_2, and $\langle c, d\rangle$*
- *Every point on X between a and b lies in the triangle bounded by L_3, L_4, and $\langle a, b\rangle$*
- *If p and q are points on X between c and d, with q closer to d, then q is closer to L_2.*
- *If p and q are points on X between b and a, with q closer to a, then q is closer to L_4.*

4 Convex hulls of multiple curves

The focus of this section is to expand our understanding of convex hulls of *sets* of taught curves. While discussing convex hulls of curves, it will be useful to clarify something that went unstated in the previous section:

Definition 10. When I refer to a *curve*, I actually mean the congruence class to a specific curve. The term "curve" refers only to a particular shape, and not a position.

When I wish to refer to a curve in a particular position, I'll mention an *instance* of a curve. If an instance of a curve is translated, rotated, or flipped, it becomes a different instance of that curve.

As a useful convention, curves will be named in capital letters, while their instances are named in lowercase letters.

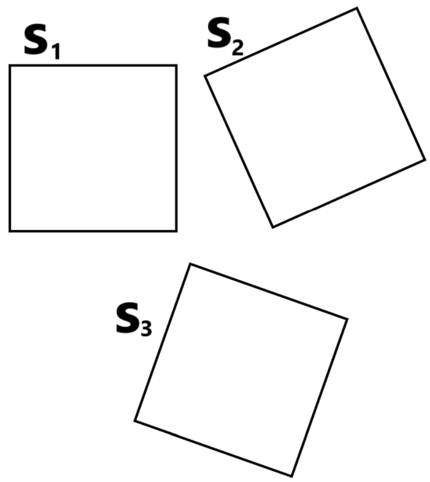


Figure 4: Three instances of S, labeled s_1, s_2, and s_3

My plan is to spend this section establishing some useful lemmas about the convex hulls of multiple curves. The nature of this problem requires that we seek to minimize the size of a convex hull, so we give a definition relating to this:

Definition 11. Given a set of curves $\{A, B, \dots\} = C$, define $[C]$ to be the set of convex hulls of unions of instances of $A, B, \dots$.

Let X be a convex hull in $[C]$. We say a X is *minimal* is it has the smallest possible area of all elements of $[C]$ convex hulls. The set of minimal convex hulls in $[C]$ will be denoted with $[C]^-$.

For convenience, we order $[C]$ by area, so that it is a weakly ordered set. We use $X \lesssim Y$ to indicate that X is smaller than or equal in size to Y. By this convention, $[C]^-$ is the set of minimal elements of $[C]$.

For a basic lemma we may use:

Lemma 8. *Let $\Omega = \{A, B, C, \dots\}$ be a set of curves, and let $\omega = \{a, b, c, \dots\}$ be a set of their instances. Let X be the convex hull of the union of ω, such that $X \in [\Omega]'$. If there is an instance of A inside X that does not touch its boundary, then $X \in [\Omega/A]^-$.*

Proof. Let a' be the instance of A that is internal to X but does not touch its boundary, let $\omega' = \{a', b, c, \dots\}$, and let X' be the convex hull of the union of ω'.

It can be immediately seen by definition that $\omega' \in \Omega$, and therefore that $X' \in [\Omega]$. Because X is a minimal element of $[\Omega]$, $X \lesssim X'$. X' contains no points outside the boundary of X, and so $X \not< X'$.

Therefore, X' has the same area as X. This implies that X is X'. $\square$

In order to produce the next lemma. we give a special definition:

Definition 12. Let X and Y be two convex shapes, and let x and y be instances of those shapes. We say that x and y are *partially intersecting* if there exists a linear translation of x, called x', such that $[y \Cup x']$ is a strict subset of $[y \Cup x']$

If two convex shapes are intersecting, but no such linear translation exists, we say those shapes are *fully intersecting*.

Lemma 9. *If x and y are two fully intersecting convex shapes, then their boundaries intersect at three or more points.*

Lemma 10. *Let Ω be a set of curves, and let $X \in [\Omega]^{-}1$ be a minimal covering set. If A and B are curves in Ω, then X contains fully intersecting instances of X and Y*

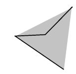
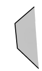
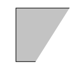
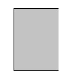
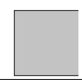
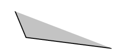
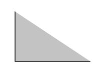
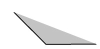
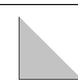
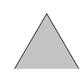
Proof. **TO BE PROVEN** □

This comes with a charming little corollary

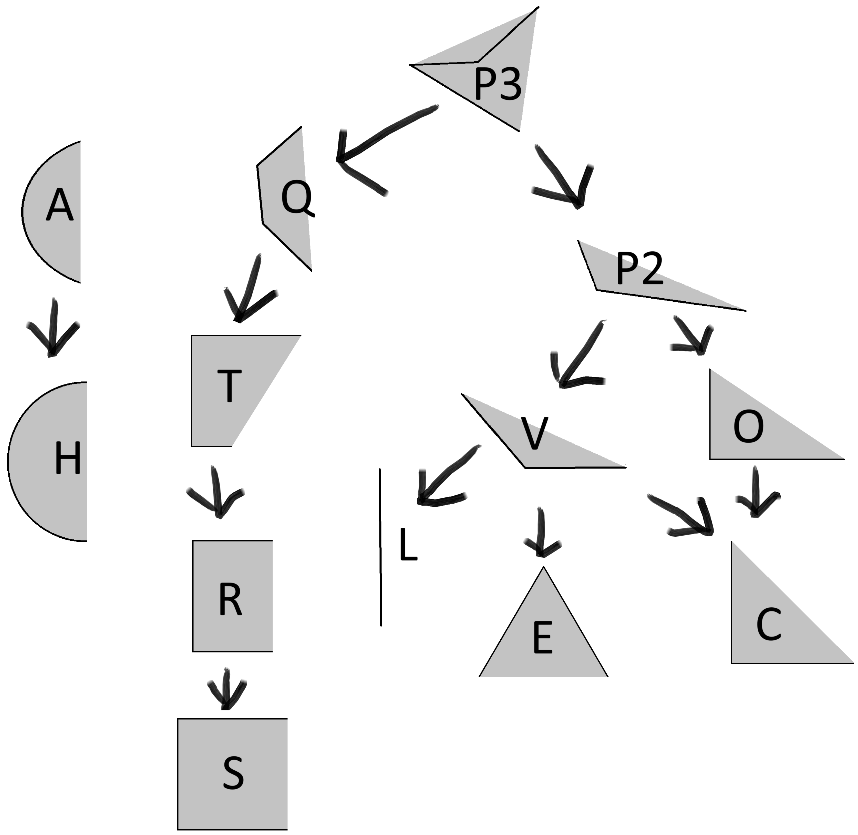
Corollary 3. *If X is a minimal cover for a set of unit length curves, no two points in X have a distance of more than 1 between them.*

Proof. **TO BE PROVEN** □

let's set up some definitions of specific curves here, so we have some toys to play with (in a manner of speaking):

Name	Description	Name justification	Illustration
$P3$	The set of 3-link polygonal chains	P for Polygon	
Q	The subset of $P3$ with clockwise non-acute angles	Q for quadrilateral	
T	The subset of $P3$ with clockwise right angles	T for Trapezoid	
R	The subset of T with equal sized first and last links	R for Rectangle	
R_w	The element of R with middle link of length w	w for width	
S	A special name for $R_{\frac{1}{3}}$	S for Square	
$P2$	The set of 2-link polygonal chains	P for Polygon	
O	The subset of $P2$ with a right angle	O for Orthogonal	
V	The subset of $P2$ with equal sized links	The letter V looks like an angle	
V_θ	The element of V with angle θ	α is used for angles	
Ω	A special name for $V_{\frac{\pi}{2}}$	Ω resembles a corner	
E	A special name for $V_{\frac{2\pi}{3}}$	E for Equilateral triangle	
A	The set of unit length circular arcs	A for arc	
A_r	The unit length arc of a circle of radius r	r for radius	
H	A special name for $A_{\frac{1}{\pi}}$	H for Hemicircle	
L	The unit line	L for line	

Here's a fun little graph to illustrate which of these are subsets of the others. The charming hand-drawn arrows indicate set inclusion.



By technicality, L is included in many of these sets, even when no arrow is drawn.

5 proxy Accommodated Curves

In a previous section, I limited the space of curves worth considering by defining taughtness.

Definition 13. a curve X is *proxy accommodated* by a set of curves Ξ if X is accommodated by every shape in $[\Xi]$.

My initial goal was to hopefully find a finite set of taut curves that proxy accommodate all other taut curves, and then to find the smallest set that contains all curves in that finite set. This goal seems reasonable at first, as it is much easier to find $[\Xi]^-$ when Ξ is a finite set of curves as opposed to an infinite one. However, as we will see, this goal is impossible, although there's a lot to be learned in trying to show this.

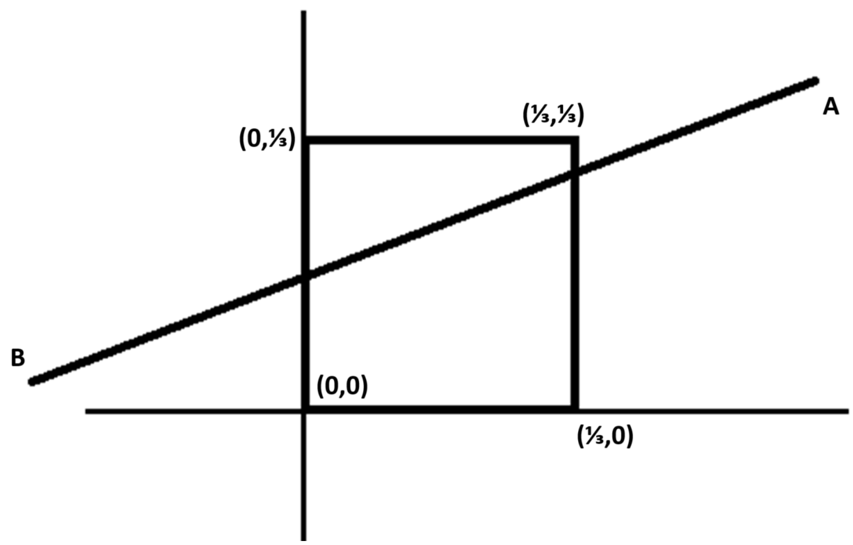
First, to illustrate that it is possible to find the minimal size for covering shapes for certain finite numbers of curves, let's give a few examples.

Lemma 11. *All elements of* $[S, L]^-$ *are congruent to a quadrilateral defined with:*

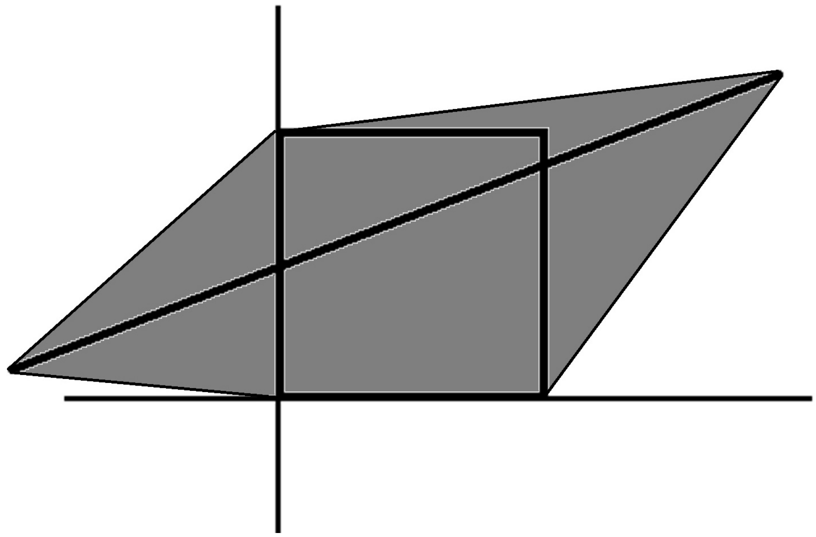
$$\left\langle \left(\frac{1}{3}, 0\right), \left(a, \frac{1}{3}\right), \left(0, \frac{1}{3}\right), (b, 0), \left(\frac{1}{3}, 0\right) \right\rangle$$

Where $b < 0$, $a > \frac{1}{3}$, *and* $a - b = \frac{2\sqrt{2}}{3}$.

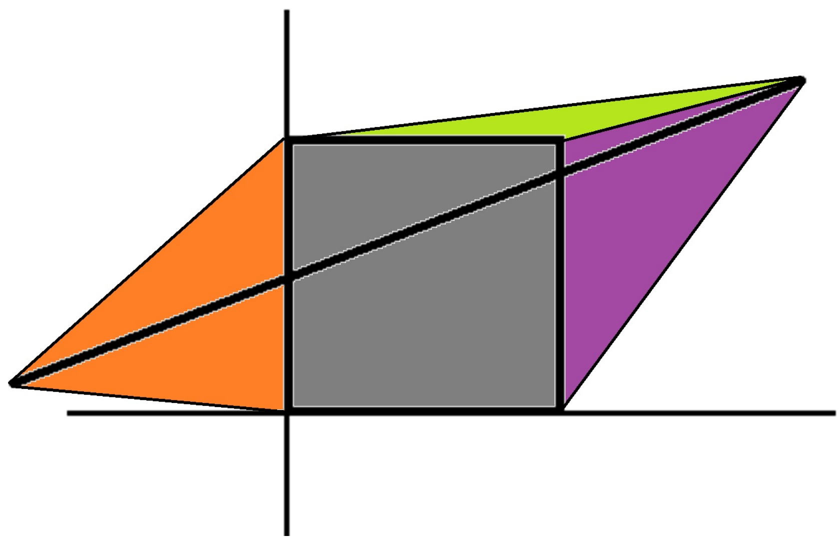
Proof. First, assume that the square is located on the points $(0, 0)$, $(0, \frac{1}{3})$, $(\frac{1}{3}, 0)$, and $(\frac{1}{3}, \frac{1}{3})$. Let the line have endpoints A and B, so that $\ell\langle A, B\rangle = 1$. Note that the line between A and B must pass through the square.



Let X be the convex hull of these points. A potential instance of X is shown in the illustration below:



X can be deconstructed into the central square and multiple triangles:



Let $A = (a_x, a_y)$, and let $B = (b_x, b_y)$. Additionally, let $f(x)$ be the distance of x to the interval $[0, \frac{1}{3}]$, so that

$$f(x) = \begin{cases} -x & x \leq 0 \\ 0 & 0 \leq x \leq \frac{1}{3} \\ x - \frac{1}{3} & \frac{1}{3} \leq x \end{cases}$$

To give an example of how this could be useful, consider the triangle formed by A_x. Using the standard formula for area of a triangle, we get $\frac{1}{2} \cdot \frac{1}{3} \cdot f(A_x)$

Then, the full area of X would be given by:

$$\frac{1}{9} + \sum_i \frac{i}{6}$$

Where i is an element of $\{A_x, A_y, B_x, B_y\}$. There is also the limitation that:

$$1 = \ell\langle A, B\rangle$$
$$1 = \sqrt{(A_x - B_x)^2 + (A_y - B_y)^2}$$
$$1 = (A_x - B_x)^2 + (A_y - B_y)^2$$

The lines $y = 0$, $y = \frac{1}{3}$, $x = 0$, and $x = \frac{1}{3}$, which define the sides of the square, also divide the plane into 9 distinct sub-regions. Those are numbered as illustrated below:

Square 4.jpg

$\square$

$$
\begin{aligned}
\left\langle \left(a, \frac{1}{3}\right), (b,0) \right\rangle &= \sqrt{(a-b)^2 + \left(\frac{1}{3}\right)^2} \\
1 &= \sqrt{(a-b)^2 + \frac{1}{9}} \\
1 &= (a-b)^2 + \frac{1}{9} \\
\frac{8}{9} &= (a-b)^2 \\
\frac{2\sqrt{2}}{3} &= a-b
\end{aligned}
$$

6 eliminating type 2 curves

So far, the previous sections have each had to do with results I have already proven. At the time that I write this, the subject of this section is not yet proven. Therefore, what follows is a hypothesis, rather than a lemma, theorem, or statement.

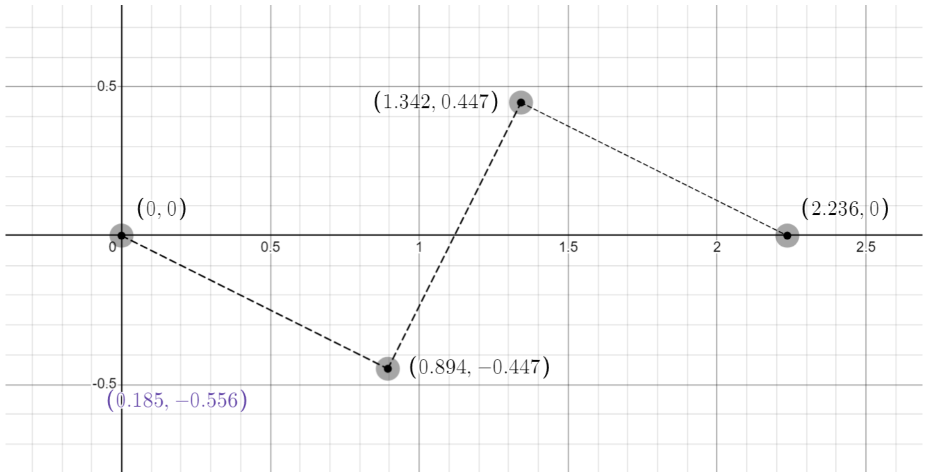
Hypothesis 1. For any Type 2 curve X, there exists a set of Type 1 curves that proxy accommodates X.

This would be a lovely result to find, although it's easier stated than proven. Let's start with a simpler hypothesis. Maybe we can use that as a stepping stone.

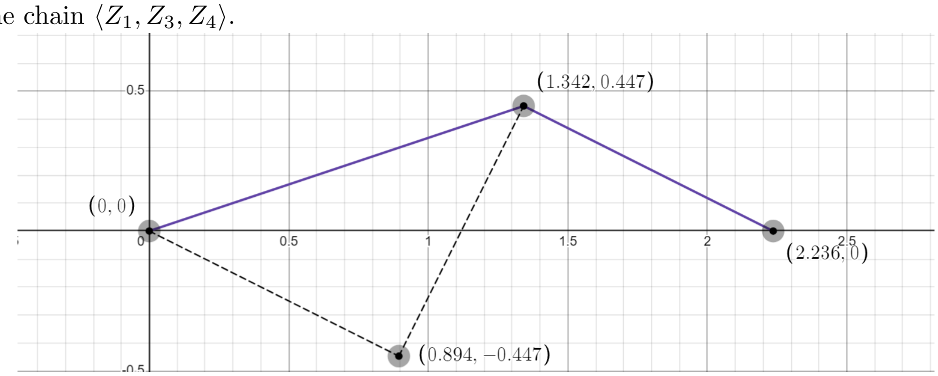
Hypothesis 2. The curve Z is proxy accommodated by a set of Type 1 curves.

To explore this, we may start by drawing the closest curve we can to Z. For convenience, let's regard all curves as having length 3, rather than length 1. We'll start with the instance of $Z = \langle Z_1, Z_2, Z_3, Z_4 \rangle$, where the points Z_1, Z_2, Z_3, and Z_4 are corners. For now, we'll set it such that:

$$
\begin{aligned}
Z_1 &= (0,0) \\
Z_2 &= (\frac{2\sqrt{5}}{5}, \frac{-\sqrt{5}}{5}) \\
Z_3 &= (\frac{3\sqrt{5}}{5}, \frac{\sqrt{5}}{5}) \\
Z_4 &= (\sqrt{5}, 0)
\end{aligned}
$$



a natural first instinct would be to create the Type 1 curve that most closely resembles this. We can start by placing a few lines on top of the Z curve, with the chain $\langle Z_1, Z_3, Z_4 \rangle$.



The purple curve here only has length $1 + /sqrt2$. Because we are allowing curves to have length 3, we may increase the purple curve's length by $2 - \sqrt{2}$. It would be wise to increase its length by the maximum possible value, and to extend it in the direction that brings the convex hull of the purple line closest to the point $(\frac{2\sqrt{5}}{5}, \frac{-\sqrt{5}}{5})$. Let P be this point.

In order to get closest to accommodating $\left(\frac{2\sqrt{5}}{5}, \frac{-\sqrt{5}}{5}\right)$, P needs to be as close as possible to the line between $\left(\frac{2\sqrt{5}}{5}, \frac{-\sqrt{5}}{5}\right)$ and the endpoint of the purple curve. That line is given by:

$$
\begin{aligned}
y - \frac{-\sqrt{5}}{5} &= \frac{\frac{\sqrt{5}}{5}}{\sqrt{5} - \frac{2\sqrt{5}}{5}} \left(x - \frac{2\sqrt{5}}{5} \right) \\
y &= \frac{1}{5-2} \left(x - \frac{2\sqrt{5}}{5} \right) - \frac{\sqrt{5}}{5} \\
y &= \frac{1}{3}x - \frac{2\sqrt{5}}{15} - \frac{3\sqrt{5}}{15} \\
y &= \frac{1}{3}x - \frac{5\sqrt{5}}{15} \\
y &= \frac{1}{3}x - \frac{\sqrt{5}}{3}
\end{aligned}
$$

The point P must be a distance of $2 - \sqrt{2}$ from the origin, and must lie on the line $y = -3x$. If $P = (P_x, P_y)$, then:

$$P_y = -3P_x$$

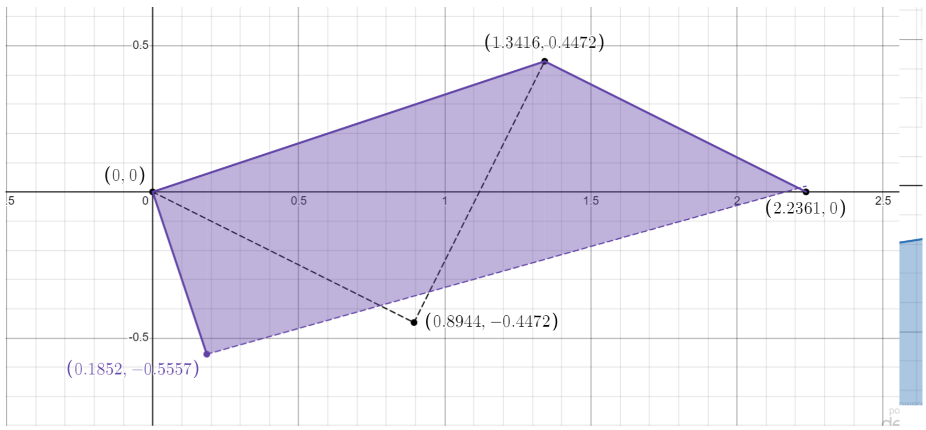
and

$$
\begin{aligned}
P_x^2 + P_y^2 &= (2 - \sqrt{2})^2 \\
P_x^2 + (-3P_x)^2 &= (2 - \sqrt{2})^2 \\
P_x^2 + 9P_x^2 &= (4 - 4\sqrt{2} + 2) \\
10P_x^2 &= 6 - 4\sqrt{2} \\
P_x^2 &= \frac{6 - 4\sqrt{2}}{10} \\
P_x &= \frac{\sqrt{3 - 2\sqrt{2}}}{\sqrt{5}}
\end{aligned}
$$

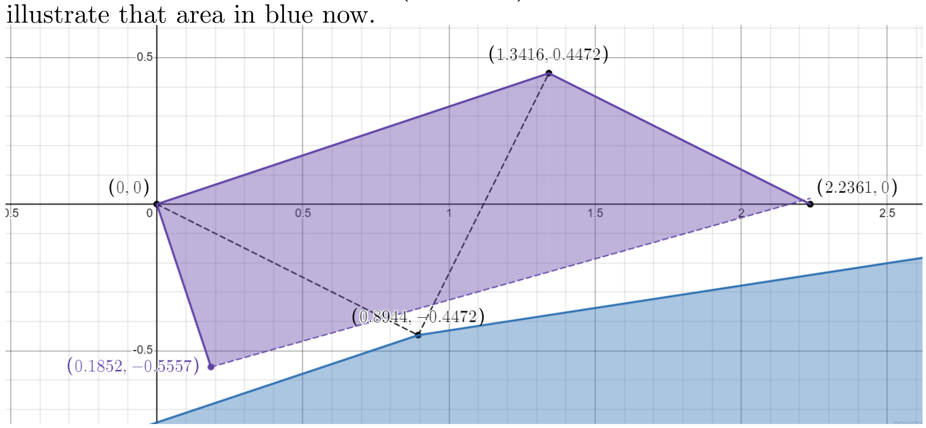
For convenience and brevity, let $u = \frac{\sqrt{3-2\sqrt{2}}}{\sqrt{5}}$. Therefore, $P = (u, -3u)$. Let's add this point to the purple curve. In total, let the purple curve be called λ, such that:

$$\lambda = \left\langle P, (0,0), \left(\frac{3\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right), \left(\sqrt{5}, 0 \right) \right\rangle$$

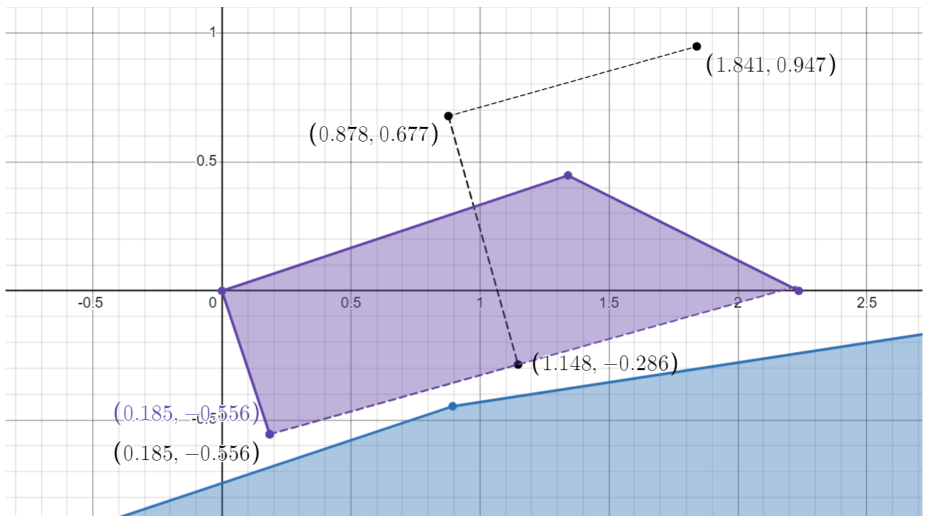
Here λ is shown in purple, with the interior shaded:



If another instance of a curve is added to this plane, that could cause Z to be accommodated. Moving forward, I'll pretend I'm trying my best to avoid accommodating Z, until I show that it's ultimately not possible. Of course, anything that would cause the point $\left(\frac{2\sqrt{5}}{5}, \frac{-\sqrt{5}}{5}\right)$ to be included is a dead zone. I'll illustrate that area in blue now.



There are other areas where the curve Z might be located that allow for it to be accommodated by other curves on the plane. for the next few steps, I'll keep the curve λ stationary, while I move the specific points in Z around. For example, Z_1 might be placed at the point P, while Z_2 lies on the boundary of $[x]$. This is illustrated here:



Finding the exact location of Z_3 and Z_4 requires knowing the slope of the line from $(u, -3u)$ to $\left(\sqrt{5}, 0\right)$. This slope is given by:

$$w = \frac{3u}{\sqrt{5} - u}$$

If (a, b) is Z_1, and the first edge of Z is at angle c, then the locations of the points in Z are:

Z_1	(a, b)
Z_2	$(a + \cos c, b + \sin c)$
Z_3	$(a + \cos c - \sin c, b + \cos c + \sin c)$
Z_4	$(a + 2\cos c - \sin c, b + \cos c + 2\sin c)$

In this case, $a = u$ and $b = -3u$. Therefore, the points in Z are:

Z_1	$(u, -3u)$
Z_2	$(u + \cos c, -3u + \sin c)$
Z_3	$(u + \cos c - \sin c, -3u + \cos c + \sin c)$
Z_4	$(u + 2\cos c - \sin c, -3u + \cos c + 2\sin c)$

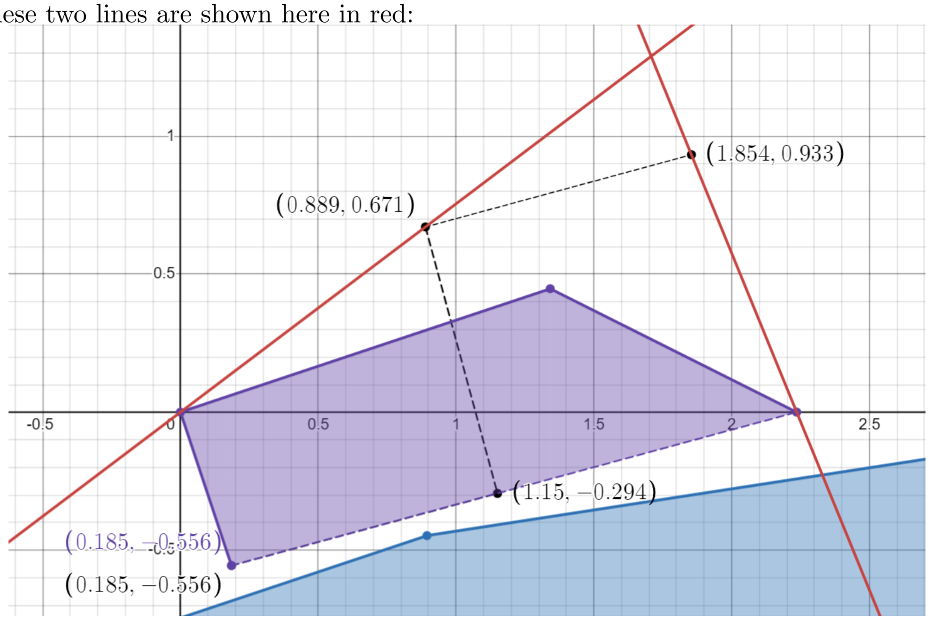
To find the point whose inclusion would accommodate the last two of these points, we find the line from the origin to the third point, then from $(\sqrt{5}, 0)$ to the last point. The first of these lines is defined by:

$$y = \frac{-3u + \cos c + \sin c}{u + \cos c - \sin c} x$$

And the second of these lines is defined by:

$$y = \frac{3u - \cos c - 2\sin c}{\sqrt{5} - (u + 2\cos c - \sin c)}\left(x - \sqrt{5}\right)$$

Since the slope of the first segment of Z is w, the angle c is equal to $\tan^{-1} w$. These two lines are shown here in red:



if the intersection of these two lines is accommodated, then the curve Z is as well. The intersection of these two lines can be found by setting them equal to each other:

$$\frac{-3u + \cos c + \sin c}{u + \cos c - \sin c}x = \frac{3u - \cos c - 2\sin c}{\sqrt{5} - (u + 2\cos c - \sin c)}\left(x - \sqrt{5}\right)$$
$$\frac{3u - \cos c - \sin c}{u + \cos c - \sin c}x = \frac{-3u + \cos c + 2\sin c}{\sqrt{5} - u - 2\cos c + \sin c}\left(x - \sqrt{5}\right)$$
$$\frac{\sqrt{5} - u - 2\cos c + \sin c}{-3u + \cos c + 2\sin c} \cdot \frac{3u - \cos c - \sin c}{u + \cos c - \sin c} = \frac{\left(x - \sqrt{5}\right)}{x}$$
$$\frac{\sqrt{5} - u - 2\cos c + \sin c}{-3u + \cos c + 2\sin c} \cdot \frac{3u - \cos c - \sin c}{u + \cos c - \sin c} = 1 - \frac{\sqrt{5}}{x}$$

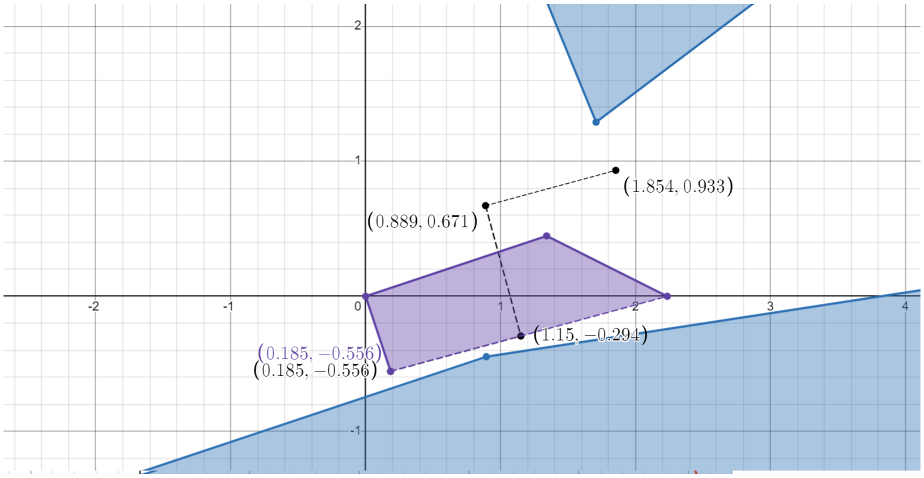
A great deal of rearranging transforms the left-hand side of this equation:

$$
\begin{aligned}
&\frac{-3u\left(\sqrt{5}-u-2\cos c+\sin c\right)+\cos c\left(\sqrt{5}-u-2\cos c+\sin c\right)+\sin c\left(\sqrt{5}-u-2\cos c+\sin c\right)}{u\left(3u-\cos c-2\sin c\right)+\cos c\left(3u-\cos c-2\sin c\right)-\sin c\left(3u-\cos c-2\sin c\right)}\\
&=\frac{3u^2-3\sqrt{5}u+\left(\sqrt{5}-4u\right)\sin c+\sin^2 c+\left(\sqrt{5}+5u\right)\cos c-2\cos^2 c-\sin c\cos c}{3u^2-5u\sin c+2\sin^2 c+2u\cos c-\cos^2 c-\sin c\cos c}\\
&=\frac{3u^2-3\sqrt{5}u+\left(\sqrt{5}-4u\right)\sin c+\sin^2 c+\left(\sqrt{5}+5u\right)\cos c-2\left(1-sin^2c\right)-\sin c\cos c}{3u^2-5u\sin c+2\sin^2 c+2u\cos c-\left(1-sin^2c\right)-\sin c\cos c}\\
&=\frac{3u^2-3\sqrt{5}u-2+2sin^2c+\left(\sqrt{5}-4u\right)\sin c+\sin^2 c+\left(\sqrt{5}+5u\right)\cos c-\sin c\cos c}{3u^2-5u\sin c+2^2c+2u\cos c-1+sin^2c-\sin c\cos c}\\
&=\frac{3u^2-3\sqrt{5}u-2+\left(\sqrt{5}-4u\right)\sin c+3\sin^2 c+\left(\sqrt{5}+5u\right)\cos c-\sin c\cos c}{3u^2-1-5u\sin c+3\sin^2 c+2u\cos c-\sin c\cos c}\\
&=\frac{3u^2-1-5u\sin c+3\sin^2 c+2u\cos c-\sin c\cos c-3\sqrt{5}u-1+\left(\sqrt{5}+u\right)\sin c+\left(\sqrt{5}+3u\right)\cos c}{3u^2-1-5u\sin c+3\sin^2 c+2u\cos c-\sin c\cos c}\\
&=1+\frac{-3\sqrt{5}u-1+\left(\sqrt{5}+u\right)\sin c+\left(\sqrt{5}+3u\right)\cos c}{3u^2-1-5u\sin c+3\sin^2 c+2u\cos c-\sin c\cos c}
\end{aligned}
$$

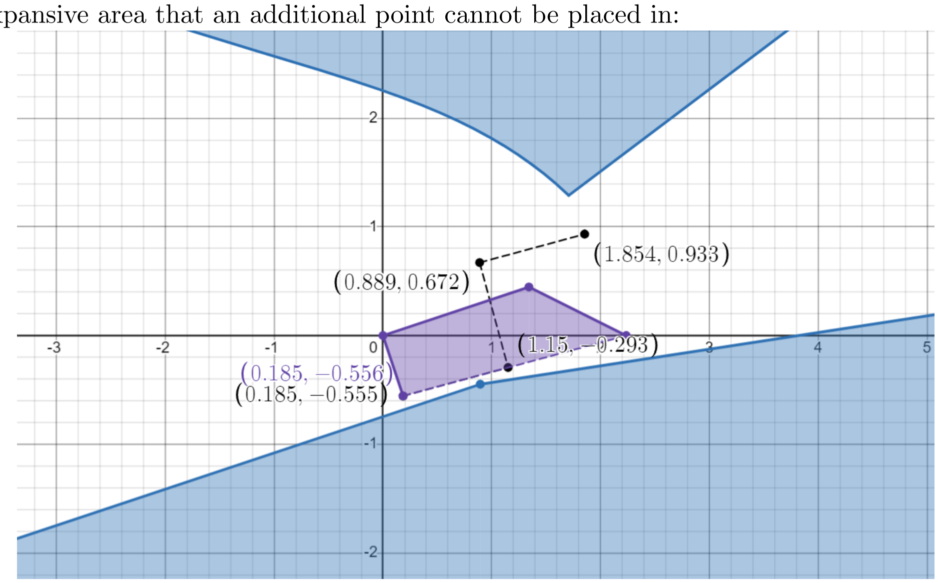
This is then inserted back into the previous equation:

$$
\begin{aligned}
1+\frac{-3\sqrt{5}u-1+\left(\sqrt{5}+u\right)\sin c+\left(\sqrt{5}+3u\right)\cos c}{3u^2-1-5u\sin c+3\sin^2 c+2u\cos c-\sin c\cos c}&=1-\frac{\sqrt{5}}{x}\\
\frac{1-3u^2+5u\sin c-3\sin^2 c-2u\cos c+\sin c\cos c}{-3\sqrt{5}u-1+\left(\sqrt{5}+u\right)\sin c+\left(\sqrt{5}+3u\right)\cos c}&=\frac{x}{\sqrt{5}}\\
\frac{1-3u^2+5u\sin c-3\sin^2 c-2u\cos c+\sin c\cos c}{-3u-\frac{1}{\sqrt{5}}+\left(1+\frac{u}{\sqrt{5}}\right)\sin c+\left(1+\frac{3u}{\sqrt{5}}\right)\cos c}&=x
\end{aligned}
$$

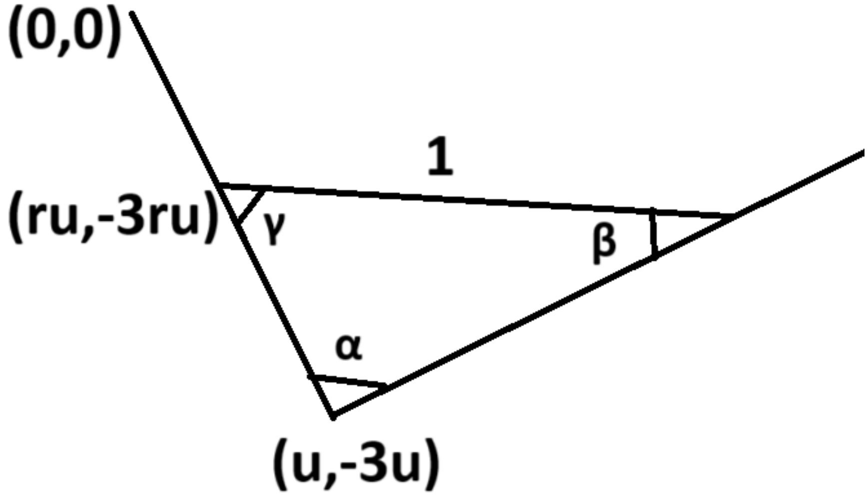
It's not pretty, but that is the x value at the intersection of those lines. As a reminder, $c=\tan^{-1} w$. Anything that accommodates that point therefore also accommodates Z. We now have an even larger blue zone:



In fact, this point was located by assuming $c = \tan^{-1} w$, but c could be any angle in the interval $[\tan^{-1} w, \frac{\pi}{2}]$. Including all these points gives an even more expansive area that an additional point cannot be placed in:



Next, let's try moving the Z curve to have Z_1 on the line from the origin to $(u, -3u)$. Let r be a number between 0 and 1 so that $Z_1 = (ru, -3ru)$ the distance from the. The rest of the points in Z can be found by finding the angle of Z's first edge. Let α, β, and γ be the angles shown below:



The distance between $(u, -3u)$ and $(ru, -3ru)$ is given by:

$$\begin{aligned}
\ell\langle(u,-3u),(ru,-3ru)\rangle &= \sqrt{\left(-3u+3ru\right)^2+\left(u-ru\right)^2} \\
&= \sqrt{9u^2-18ru^2+9r^2u^2+u^2-2ru^2+r^2u^2} \\
&= u\sqrt{9-18r+9r^2+1-2r+r^2} \\
&= u\sqrt{10r^2-20r+10} \\
&= u\sqrt{10\left(r^2-2r+1\right)} \\
&= u\sqrt{10\left(r-1\right)^2} \\
&= u\left(r-1\right)\sqrt{10}
\end{aligned}$$

The slope of the line from $(0, 0)$ to $(u, -3u)$ is -3, and therefore its angle is $\tan^{-1}(-3)$. The slope of the line from $(u, -3u)$ to $(0, \sqrt{5})$ is w, and therefore its angle is $\tan^{-1}(w)$. In total therefore, $\alpha = \tan^{-1}(w) - \tan^{-1}(-3)$. At this point now, we can use the law of sines to find

$$\begin{aligned}
\frac{\sin\beta}{u\left(r-1\right)\sqrt{10}} &= \frac{\sin\alpha}{1} \\
\sin\beta &= u\left(r-1\right)\sqrt{10}\sin\alpha \\
\beta &= \sin^{-1}\left(u\left(r-1\right)\sqrt{10}\sin\alpha\right)
\end{aligned}$$

Therefore, γ can be found as well:

$$\begin{aligned}\gamma &= 2\pi - \alpha - \beta \\ &= 2\pi - \alpha - \sin^{-1}\left(u\,(r-1)\,\sqrt{10}\sin\alpha\right)\end{aligned}$$

Finally, the angle c of the line from Z_1 to Z_2 is given by:

$$c = \tan^{-1}(-3) - 2\pi + \alpha + \sin^{-1}\left(u\,(r-1)\,\sqrt{10}\sin\alpha\right)$$

For simplicity, this angle is coterminal with:

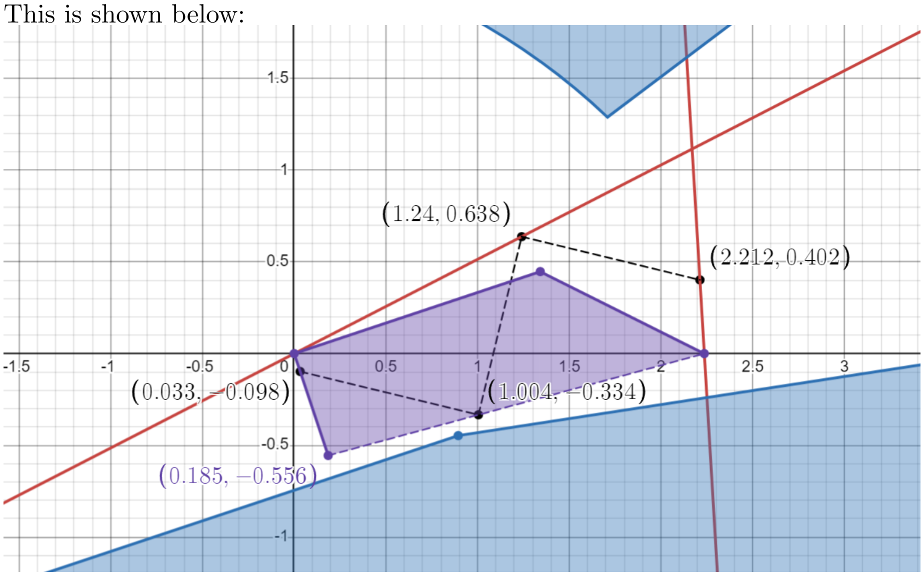
$$c = \tan^{-1}(w) + \sin^{-1}\left(u\,(r-1)\,\sqrt{10}\sin\alpha\right)$$

the lines from λ to the points Z_1 and Z_2 are then given by:

$$y = \frac{-3ru + \cos c + \sin c}{ru + \cos c - \sin c}x$$

$$y = \frac{-3ru + \cos c + 2\sin c}{ru + 2\cos c - \sin c - \sqrt{5}}\left(x - \sqrt{5}\right)$$

This is shown below:



As before, These are set equal to each other in order to find the x coordinate of the intersection:

$$\frac{-3ru + \cos c + \sin c}{ru + \cos c - \sin c}x = \frac{-3ru + \cos c + 2\sin c}{ru + 2\cos c - \sin c - \sqrt{5}}\left(x - \sqrt{5}\right)$$

Again, we work towards solving for x:

$$\frac{ru + 2\cos c - \sin c - \sqrt{5}}{-3ru + \cos c + 2\sin c}\frac{-3ru + \cos c + \sin c}{ru + \cos c - \sin c} = \frac{\left(x - \sqrt{5}\right)}{x}$$

The left hand side of this equation expands and simplifies to:

$$\frac{-3r^2u^2 - 5ru\cos c + 4ru\sin c + 3\sqrt{5}ru + 2\cos^2 c - \sqrt{5}\cos c + \cos c\sin c - \sin^2 c - \sqrt{5}\sin c}{-3r^2u^2 + 5ru\sin c - 2ru\cos c + \cos^2 c + \sin c\cos c - 2\sin^2 c}$$

By subtracting the denominator from the numerator, this becomes:

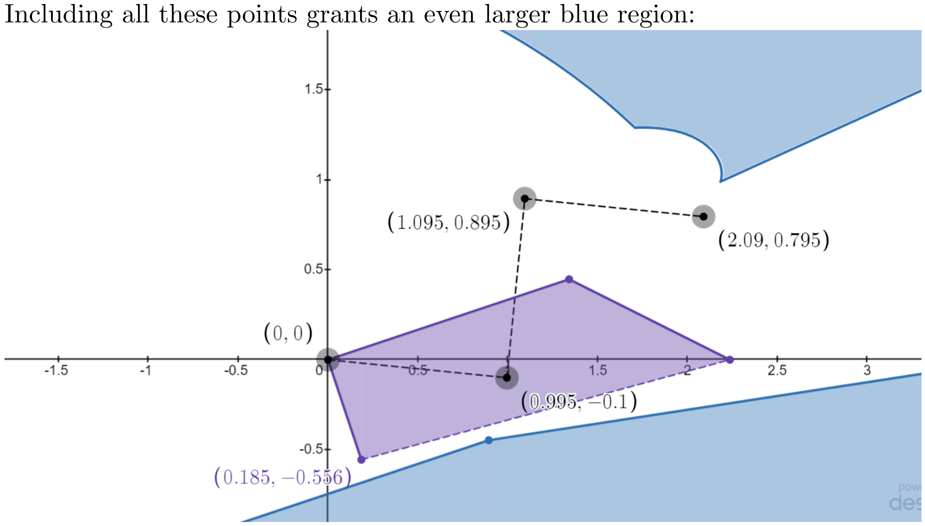
$$1 + \frac{-3ru\cos c - ru\sin c + 3\sqrt{5}ru + \cos^2 c - \sqrt{5}\cos c + \sin^2 c - \sqrt{5}\sin c}{-3r^2u^2 + 5ru\sin c - 2ru\cos c + \cos^2 c + \sin c\cos c - 2\sin^2 c}$$

And finally, therefore:

$$\frac{-3ru\cos c - ru\sin c + 3\sqrt{5}ru + \cos^2 c - \sqrt{5}\cos c + \sin^2 c - \sqrt{5}\sin c}{-3r^2u^2 + 5ru\sin c - 2ru\cos c + \cos^2 c + \sin c\cos c - 2\sin^2 c} = -\frac{\sqrt{5}}{x}$$

$$\sqrt{5}\frac{3r^2u^2 - 5ru\sin c + 2ru\cos c - \cos^2 c - \sin c\cos c + 2\sin^2 c}{-3ru\cos c - ru\sin c + 3\sqrt{5}ru + \cos^2 c - \sqrt{5}\cos c + \sin^2 c - \sqrt{5}\sin c} = x$$

Including all these points grants an even larger blue region:



Although there is a large amount of work that can still be done to limit the area, I'll set that down for the time being. I've got a second plan for how this can be approached, which hopefully will be less grueling. I might write more here later!

6.1 second attempt

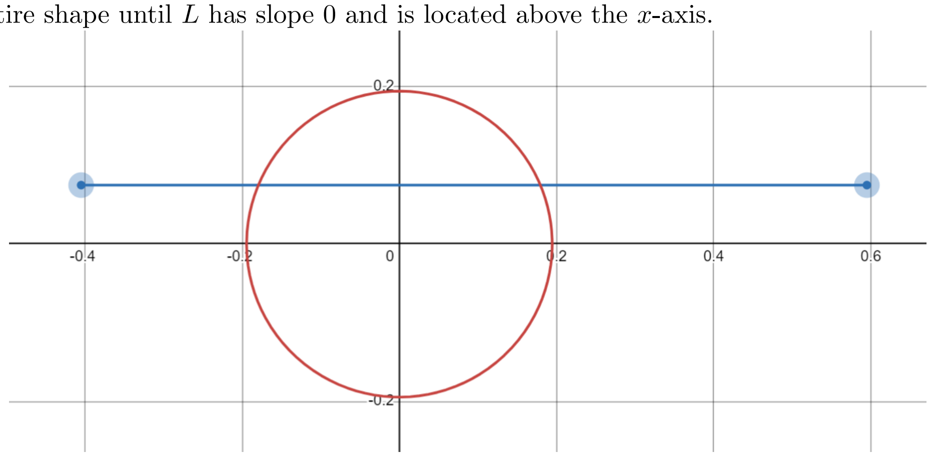
First and foremost, we're going back to considering curves to have a length of 1. I don't know why I thought changing the length of curves to 3 would simplify things. Here's the first piece of inspiration that leads to this new approach:

Lemma 12. *The largest circle that can be accommodated by the convex hull of a single unit-length curve has a radius of* $\frac{1}{\pi+2}$.

Proof. **TO BE PROVED** $\square$

For brevity, let's call this curve M. This circle might not seem especially large, but when compared with the unit length circle (radius $= \frac{1}{2\pi}$) or the circle circumscribed inside S (radius $= \frac{1}{6}$), this is a worthwhile improvement.

In any minimal convex hull with M and L, L must pass through the entirety of M. Without loss of generality, place a circle of radius $\frac{1}{\pi+2}$ centred at the origin, called m. Allow a line of unit length to pass through it, and rotate the entire shape until L has slope 0 and is located above the x-axis.



Precisely defining the convex hull of these two curves is cumbersome, but doable. First, the location of L can be uniquely determined by its left point. Define this point at (a, b). For convenience, let r be $\frac{1}{\pi+2}$. The central circle can be defined with

$$y^2 + x^2 = r^2$$

The derivative is then given by:

$$\begin{aligned} 2yy' + 2x &= 0 \\ y' &= -\frac{x}{y} \end{aligned}$$

Let (x_1, y_1) be the point on the upper half of the circle whose tangent line passes through (a, b). This gives:

$$
\begin{aligned}
y' &= \frac{\Delta y}{\Delta x} \\
\frac{-x_1}{y_1} &= \frac{y_1 - b}{x_1 - a} \\
-x_1^2 + a x_1 &= y_1^2 - b y_1 \\
b y_1 + a x_1 &= y_1^2 + x_1^2 \\
b y_1 + a x_1 &= r^2 \\
y_1 &= \frac{r^2 - a x_1}{b}
\end{aligned}
$$

Because this point is located on the upper half of the circle, we may assume $y_1 = \sqrt{r^2 - x_1^2}$, and therefore:

$$
\begin{aligned}
\sqrt{r^2 - x_1^2} &= \frac{r^2 - a x_1}{b} \\
r^2 - x_1^2 &= \frac{r^4 - 2ar^2 x_1 + a^2 x_1^2}{b^2} \\
b^2 r^2 - b^2 x_1^2 &= r^4 - 2ar^2 x_1 + a^2 x_1^2 \\
0 &= r^4 - 2ar^2 x_1 + a^2 x_1^2 + b^2 x_1^2 - b^2 r^2 \\
0 &= \left(a^2 + b^2\right) x_1^2 - 2ar^2 x_1 + \left(r^4 - b^2 r^2\right)
\end{aligned}
$$

This can be solved with the quadratic formula:

$$
\begin{aligned}
x_1 &= \frac{2ar^2 \pm \sqrt{\left(-2ar^2\right)^2 - 4\left(a^2 + b^2\right)\left(r^4 - b^2 r^2\right)}}{2\left(a^2 + b^2\right)} \\
x_1 &= \frac{2ar^2 \pm \sqrt{4a^2 r^4 - 4\left(a^2 r^4 - a^2 b^2 r^2 + b^2 r^4 - b^4 r^2\right)}}{2\left(a^2 + b^2\right)} \\
x_1 &= \frac{ar^2 \pm \sqrt{a^2 r^4 - a^2 r^4 + a^2 b^2 r^2 - b^2 r^4 + b^4 r^2}}{a^2 + b^2} \\
x_1 &= r\frac{ar \pm b\sqrt{a^2 + b^2 - r^2}}{a^2 + b^2}
\end{aligned}
$$

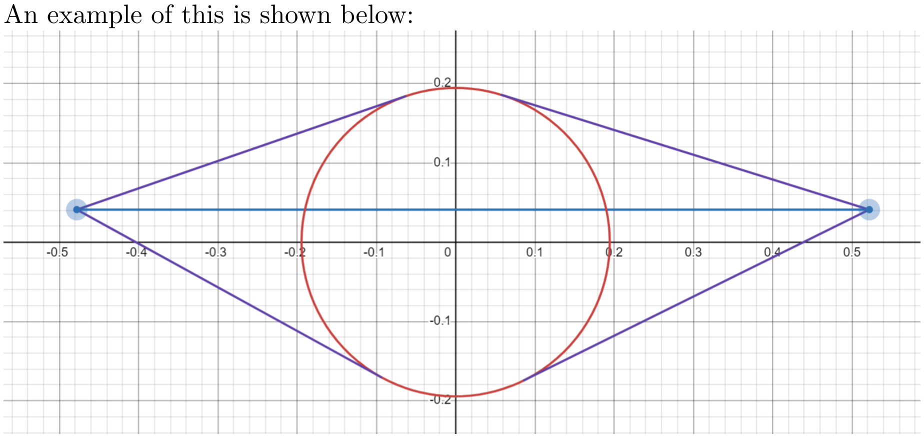
Because (a, b) is located above the x-axis, we will use the positive version of this expression. We may repeat this process to find the other points:

$x_1 = r\frac{ar+b\sqrt{a^2+b^2-r^2}}{a^2+b^2}$	$y_1 = \sqrt{r^2 - d_1^2}$
$x_2 = r\frac{ar-b\sqrt{a^2+b^2-r^2}}{a^2+b^2}$	$y_2 = -\sqrt{r^2 - d_2^2}$
$x_3 = r\frac{(a+1)r+b\sqrt{(a+1)^2+b^2-r^2}}{(a+1)^2+b^2}$	$y_3 = -\sqrt{r^2 - d_3^2}$
$x_4 = r\frac{(a+1)r-b\sqrt{(a+1)^2+b^2-r^2}}{(a+1)^2+b^2}$	$y_4 = \sqrt{r^2 - d_4^2}$

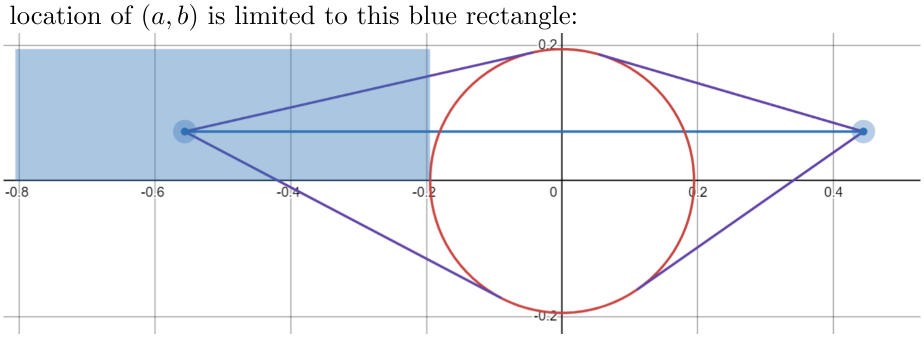
Using these points and the slopes associated with them, the four lines outlining the convex hull are:

$y_1 = -\frac{x_1}{\sqrt{r^2-x_1^2}}(x-a)+b$	$x \in [a, x_1]$
$y_2 = \frac{x_2}{\sqrt{r^2-x_2^2}}(x-a)+b$	$x \in [a, x_2]$
$y_3 = \frac{x_3}{\sqrt{r^2-x_3^2}}(x-a-1)+b$	$x \in [x_3, a+1]$
$y_4 = -\frac{x_4}{\sqrt{r^2-x_4^2}}(x-a-1)+b$	$x \in [x_4, a+1]$

An example of this is shown below:



As mentioned, we assume without loss of generality that the line here is located entirely above the x-axis and that it passes entirely through m. Therefore, the location of (a, b) is limited to this blue rectangle:

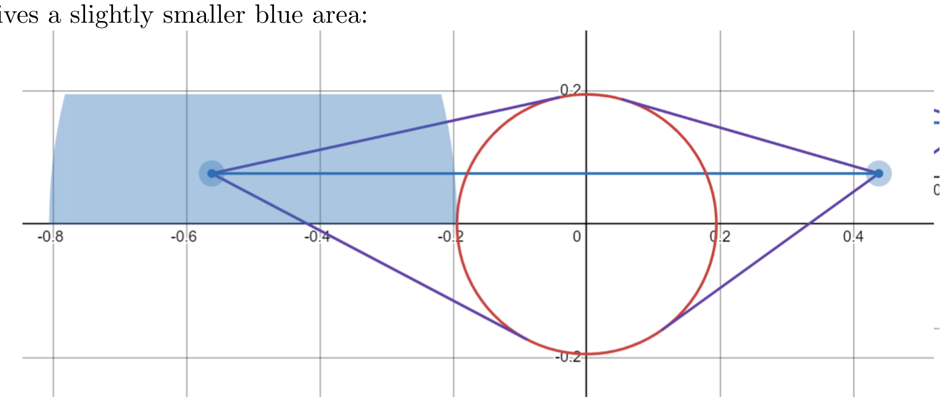


by Corollary 3, the distance from (a, b) to the furthest point of the circle is 1.

The point (a, b) has a distance from the origin of $\sqrt{a^2+b^2}$. Therefore, wherever (a, b) is located, it must satisfy the inequality

$$\sqrt{a^2+b^2}+r\leq 1$$

This inequality intersects $y=r$ where $x=-\sqrt{1-2r}$, resulting in a small chip off the available area for (a,b). Applying the same principle to $(a+1,b)$ gives a slightly smaller blue area:



We also wish to eliminate all points in this blue region that result in a convex hull that accommodates Z. For a start, let's define the first point of Z, which we call Z_1, to be (a_1,b_1), and let the first edge of Z exist at an angle of c. Then the points in Z are defined with:

Z_1	(a_1,b_1)
Z_2	$\left(a_1+\frac{\cos c}{3},b_1+\frac{\sin c}{3}\right)$
Z_3	$\left(a_1+\frac{\cos c-\sin c}{3},b_1+\frac{\cos c+\sin c}{3}\right)$
Z_4	$\left(a_1+\frac{2\cos c-\sin c}{3},b_1+\frac{\cos c+2\sin c}{3}\right)$

Let's start by assuming that Z_1 and Z_2 lie on y_2. This gives that:

$$a_1=a+(x_2-a)\,p$$

$$b_1=b-\left(b+\sqrt{r^2-x_2^2}\right)p$$

$$c=\tan^{-1}\frac{x_2}{\sqrt{r^2-x_2^2}}$$

Where p is some value above 0. Z_2 will lie on y_2 as long as it's x coordinate is within the boundary set earlier. In other words:

$$a_1+\cos c\leq x_2$$

$$a+(x_2-a)\,p+\cos\left(\tan^{-1}\frac{x_2}{\sqrt{r^2-x_2^2}}\right)\leq x_2$$

There are two other requirements for Z to be accommodated by this shape. First, Z_3 must be included. Because Z_3 is exactly 1/3 units away from y_2, this only requires that Z_3 is below y_1. Second, Z_4 must be included, which requires both that Z_4 lies above y_3 and that Z_4 lies below y_4. All told, the four requirements described so far are:

1. Z_2 lies on y_2
2. Z_3 lies on or under y_1
3. Z_4 lies on or under y_4
4. Z_4 lies on or above y_3

Naturally, the first of these places an upper bound on p. Because increasing p slides Z_4 in the direction of y_2, and y_3 has a positive slope, the fourth requirement also places an upper bound on p. The third requirement places a lower bound on p when the slope of y_2 is greater than the slope of y_4, but experimentally, it is a weaker bound than the lower bound made by the second requirement. For this reason, we'll simply assume Z_3 lies on y_1, and calculate p based on this.

The calculations involved here are fairly complicated, so

7 Walk on a Grid

Dr. Kovchegov has assured me that this is a bad approach to this problem, and that this problem should be thought of analytically. However, Yevgeniy is not the math police, and everyone who has attempted an analytical approach has so far failed. So here's a genuinely insane angle to approach this problem. If you had a polygonal chain where each angle is a right angle and each link has the same length, that would be kind of like a walking on a grid, right?

Problem 5. *For any value of n, What is the smallest grid that can accommodate any self-avoiding walk with n steps?*

8 Analytical Approach